Let $M^n$ be a Riemannian manifold of dimension $n \geq 2$ and class $C^3$, $(g_{ij})$ the symmetric matrix of the positive definite metric of $M^n$, and $(g^{ij})$ the inverse matrix of $(g_{ij})$, and denote by $\nabla_i$, $R_{ijkl}$, $R_{ij} = R^k_{ijk}$ and $R = g^{ij}R_{ij}$ the operator of covariant differentiation with respect to $g_{ij}$, the Riemann tensor, the Ricci tensor and the scalar curvature of $M^n$ respectively. Throughout this paper all Latin indices take the values $1, \ldots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using $g^{ij}$ and $g_{ij}$ respectively, and that repeated indices imply summation.

Let $v$ be a vector field defining an infinitesimal conformal transformation on $M^n$. Denote by the same symbol $v$ the 1-form corresponding to the vector field $v$ by the duality defined by the metric of $M^n$, and by $Lv$ the operator of the infinitesimal transformation $v$. Then we have

\[(1.1) \quad L_v g_{ij} = \nabla_i v_j + \nabla_j v_i = 2pg_{ij}.\]

The infinitesimal transformation $v$ is said to be homothetic or an infinitesimal isometry according as the scalar function $p$ is constant or zero. We also denote by $Ldp$ the operator of the infinitesimal transformation generated by the vector field $p^i$ defined by

\[(1.2) \quad p^i = g^{ij}p_j, \quad p_j = \nabla_j p.\]

Let $\xi_1 \ldots i_p$ and $\eta_1 \ldots i_p$ be two tensor fields of the same order $p \leq n$ on a compact orientable manifold $M^n$. Then the local and global scalar products $\langle \xi, \eta \rangle$ and $\langle \xi, \eta \rangle$ of the tensor fields $\xi$ and $\eta$ are defined by

\[(1.3) \quad \langle \xi, \eta \rangle = \frac{1}{p!} \xi^{i_1 \ldots i_p} \eta_{i_1 \ldots i_p}.\]
\[(1.4) \quad \langle \xi, \eta \rangle = \int_{M^n} \langle \xi, \eta \rangle dV,\]

where \(dV\) is the element of volume of the manifold \(M^n\) at a point.

In the last decade or so various authors have studied the conditions for a Riemannian manifold \(M^n\) of dimension \(n>2\) with constant scalar curvature \(R\) to be either conformal or isometric to an \(n\)-sphere. Very recently Yano, Obata, Hsiung and Mugridge (see [6], [4], [2]) have been able to extend some of the above-mentioned results by replacing the constancy of \(R\) by \(L_vR=0\), where \(u\) is a certain vector field on \(M^n\). The purpose of this paper is to continue their work by establishing the following theorems.

To begin we denote by (C) the following condition:

A compact Riemannian manifold \(M^n\) of dimension \(n>2\)
(C) admits an infinitesimal nonisometric conformal transformation \(v\) satisfying (1.1) with \(\rho \neq 0\) such that \(L_vR=0\).

**Theorem I.** An orientable \(M^n\) is conformal to an \(n\)-sphere if it satisfies condition (C) and

\[(1.5) \quad \left( \rho \rho \rho^i - \frac{1}{n-1} R\rho^2, R \right) \geq 0,\]

\[(1.6) \quad L_v \left( a^2 A + \frac{c - 4a^2}{n-2} B \right) = 0,\]

where \(A\) and \(B\) are defined by

\[(1.7) \quad A = R^{hijk} R_{hijk}, \quad B = R^{ij} R_{ij},\]

and \(a, c\) are constant such that

\[(1.8) \quad c = 4a^2 + (n-2) \left[ 2a \sum_{i=1}^{4} b_i + \left( \sum_{i=1}^{6} (-1)^{i-1} b_i \right)^2 \right.\]
\[-2(b_1b_3 + b_2b_4 - b_3b_5) + (n-1) \sum_{i=1}^{6} b_i^2 \biggr] > 0,\]

\(b\)'s being any constants.

For the case \(a \neq 0, c - 4a^2 = 0\) and the case \(a = 0, c - 4a^2 \neq 0\), Theorem I is due to Yano [4].

**Theorem II.** A manifold \(M^n\) is conformal to an \(n\)-sphere, if it

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*An elementary calculation shows that \(c \geq 0\) where equality holds if and only if \(b_1 = \cdots = b_4, b_5 = b_6 = 0, a = -(n-2)b_1\).*
satisfies condition (C) and any one of the following three sets of conditions:

(1.9) \[ \nabla_i \nabla_j (Rf) = R g_{ij} \quad (f \text{ is a scalar function}), \]

(1.10) \[ Q \varphi = \frac{2}{n} \varphi (R \varphi), \quad \nabla_i \nabla_j (R \varphi) = R \nabla_i \nabla_j \varphi, \]

(1.11) \[ L \alpha g_{ij} = \alpha g_{ij} \quad (\alpha \text{ is a scalar function}), \]

where \( Q \) is the operator of Ricci defined by, for any vector field \( u \) on \( M^n \),

(1.12) \[ Q: u_i \rightarrow 2 R_{ij} u^j. \]

For constant \( R \), conditions (1.10) and (1.11) in Theorem II will lead to the conclusion that \( M^n \) is isometric to an \( n \)-sphere of radius \( (n(n-1)/R)^{1/2} \); for this see [5].

**Theorem III.** A manifold \( M^n \) with constant \( R \) is isometric to an \( n \)-sphere of radius \( (n(n-1)/R)^{1/2} \), if it satisfies conditions (C) and (1.9).

Theorem III is due to Lichnerowicz [3] when condition (1.9) is replaced by the following one:

(1.13) \( v \) is the gradient of a scalar function \( f \), i.e., \( v_i = \nabla_i f \).

For constant \( R \), it is easily seen that condition (1.13) is a special case of condition (1.9). In fact, in this case by using (1.2) condition (1.9) becomes \( \nabla_i v_j + \nabla_j v_i = 2 \nabla_i \nabla_j f \), which is satisfied by \( v_i = \nabla_i f + u_i \) where \( u_i \) is any vector field generating an infinitesimal isometry.

**Theorem IV.** A manifold \( M^n \) is isometric to an \( n \)-sphere, if it satisfies condition (C), \( L \varphi R = 0 \), and

(1.14) \[ A^a B^b = \epsilon = \text{const}, \]

(1.15) \[ c \left( \frac{2a}{A} + \frac{(n - 1)b}{B} \right) = \frac{2^a(a + b) R^2(a+b-1)}{n^{a+b-1}(n-1)^{a-1}}, \]

where \( A, B \) are given by (1.7), and \( a, b \) are nonnegative integers and not both zero.

For constant \( R \), Theorem IV is due to Lichnerowicz [3] for \( a = 0 \), \( b = 1 \) and due to Hsiung [1] for general \( a \) and \( b \).

**Bibliography**


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