THE VITALI-HAHN-SAKS AND NIKODYM THEOREMS
FOR ADDITIVE SET FUNCTIONS

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ABSTRACT. The purpose of this note is twofold. Firstly, we point out an appropriate version of the Vitali-Hahn-Saks and Nikodym theorems for finitely additive set functions defined on a sigma algebra of sets. Secondly, we apply our Theorem to extend a recent result of James K. Brooks for countably additive vector valued set functions to the general finitely additive case.

THEOREM. Let \( \{\mu_n\} \) be a sequence of bounded and finitely additive scalar valued set functions defined on a sigma algebra \( \Sigma \) of subsets of a set \( S \). If \( \mu(E) = \lim_{n \to \infty} \mu_n(E) \) exists for every \( E \subseteq \Sigma \), then \( \mu \) is bounded and additive and the additivity of the \( \mu_n \) is uniform in \( n \).

In addition, suppose \( \lim_{E \to 0} \mu_n(E) = 0 \) for each \( n \), where \( \nu \) is a nonnegative finitely additive set function defined on \( \Sigma \). Then \( \lim_{E \to 0} \nu_n(E) = 0 \) uniformly in \( n \).

Our Theorem follows from the weak convergence theory for finitely additive set functions. While a proof of it can be synthesized (modulo an observation) from [3], [5], [6] and [7], for the readers' convenience we shall also refer to the discussion of the work of Soloman Leader and Pasquale Porcelli [6] and [7] given in [4]. The corollary on p. 475 of [3] tells us that the sequence \( \{\mu_n\} \) is weakly convergent. Also, the \((L)\)-space of bounded and additive functions on \( \Sigma \) is weakly complete [5, Theorem 12], so the sequence \( \{\mu_n\} \) converges weakly to \( \mu \). Hence \( \mu \) is bounded and additive, and (cf. [4]) the weakly convergent sequence \( \{\mu_n\} \) is equi-absolutely continuous with respect to the bounded and additive function \( \varphi \) defined on \( \Sigma \) by

\[
\varphi(E) = \sum_{k=1}^{\infty} 2^{-k} (1 + |\mu_k|)(S)^{-1} |\mu_k| \langle E \rangle,
\]

where \( |\mu_k| \) is the variation of \( \mu_k \). Since the sequence \( \{\mu_n - \mu\} \) converges weakly to zero, Lemma 1 of [4] implies

\[
\lim_{J} \left\{ \sup_{k} \left[ \sum_{i,j} |\mu_k(E_{ij}) - \mu(E_{ij})| \right] \right\} = 0
\]
whenever \( \{E_i\} \) is a sequence of pairwise disjoint elements of \( \Sigma \). Moreover, because \( \mu \) is bounded and additive, \( \sum \mu(E_i) \leq |\mu| < \infty \). Hence

\[
\lim_j \left\{ \sup_k \left[ \sum |\mu_k(E_i)| \right] \right\} = 0
\]

whenever \( \{E_i\} \) is a sequence of pairwise disjoint elements of \( \Sigma \). In the countably additive case, uniform countable additivity is equivalent to (3). Thus (3) represents a reasonable definition of uniform additivity for the sequence \( \{\mu_n\} \). Finally, if each \( \mu_n \) is absolutely continuous with respect to \( \nu \), then \( \varphi \) is absolutely continuous with respect to \( \nu \).

Suppose that \( \mathfrak{B} \) is a separable Banach space over the complex numbers.

**Corollary.** Let \( \mu_n \) be a sequence of finitely additive \( \mathfrak{B} \)-valued set functions defined on \( \Sigma \) such that \( \lim_n \mu_n(E) \) exists for every \( E \in \Sigma \). Suppose \( \lim_{n \to 0} \mu_n(E) = 0 \) for each \( n \), where \( \nu \) is a nonnegative (real valued) finitely additive set function defined on \( \Sigma \). Then \( \lim_{n \to 0} \mu_n(E) = 0 \) uniformly in \( n \).

The proof given by Brooks in [1] for the countably additive version of this Corollary carries over if our Theorem is used and one notices that the \( \mu_n \) of the Corollary is bounded and additive on \( \Sigma \) by Theorem 3.2 of [2].

**References**


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