THE ORDER OF THE IMAGE OF THE J-HOMOMORPHISM

BY MARK MAHOWALD

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ABSTRACT. This note announces a proof of the order of the image of the J-homomorphism and gives several other results in homotopy theory which are consequences of the proof.

The set $\Omega^n S^n$ can be identified with the set of all base point preserving maps of S^n into itself. $SO(n)$, acting on S^n as R^n with a point at infinity, is also a set of base point preserving maps of S^n onto itself. This defines $SO(n) \subset \Omega^n S^n$. The induced map in homotopy is called the J-homomorphism. If we allow n to go to infinity we have the stable J-homomorphism. By Bott's results [3] $\pi_j(SO) = Z$, $j \equiv -1 \bmod 4$, and $= Z_2 j \equiv 0, 1 \bmod 8$, $j > 0$, and zero otherwise. Adams [1] showed that the Z_2 summand maps monomorphically and Milnor and Kervaire [6] showed that the Z group in dimension $4j-1$ maps nontrivially and its image generates a subgroup of at least a certain order λ_j'. Adams [1] showed that the order was either λ_j' or $2\lambda_j'$ and if $j \equiv 1$ (2) it was λ_j'. Thus only the two primary part is in question and there only for $j \equiv 0$ (2). Let λ_j be the two primary part of λ_j'. If $4j \equiv 2^{\rho(j)} \bmod 2^{\rho(j)+1}$ (which defines $\rho(j)$) then $\lambda_j = 2^{\rho(j)+1}$. We prove:

THEOREM 1. *The* 2-*primary order of the image of* J *in stem* $4j-1$ *is* λ_j.

The proof has several corollary results which have some interest. The first result is rather technical but still has some interest. The naming of elements in $H^{**}(A)$ is that given in [5].

THEOREM 2. *The elements* $P^i c_0$, $P^i h_1 c_0$, $i \geqq 1$, $P^i h_2$, $i \geqq 1$, *in* $H^{**}(A)$ *represent the image of* J *in dimension* $j \equiv 0, 1, 3 \bmod 8$. *In dimension* $8j-1$ *the "tower" which ends at the "Adams edge" represents the image of* J *in that dimension.*

These elements were known to have the desired e-invariant property [1] and were believed to be in J. Their Whitehead product behavior has been investigated ([2] and [4], for example).

Let $M = Z_2 + Z_2$ (be the module over A with one generator; μ in

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dimension zero and $Sq^1 \neq 0$. Let $P(x_1, \cdots)$ be a polynomial algebra on generators x_i with bidegree $(2, 2^{i+2}+1)$. Consider the differential $d(x_i) \to x_{i-1}^2 x_1$ in P. Let $H(d)$ be the homology under d and $B(d) = \text{im } d$.

For $\alpha \in P$ let the bidegree of α be (α_s', α_t'). We will be only interested in the values of α_s' modulo 4 and α_t' modulo 12 so take (α_s, α_t) so that $\alpha_s \equiv \alpha_s'$ (mod 4), $\alpha_t \equiv \alpha_t'$ (mod 4) but $5\alpha_s < \alpha_t - \alpha_s$.

THEOREM 3. *If* $5s \geqq t-s+\epsilon$ *where* ϵ *depends on the congruence class of* s mod 4 *and* $\epsilon \leqq 6$, *then*

$$\operatorname{Ext}_A^{s,t}(M, Z_2) = \sum_{\alpha \in H(d)} \operatorname{Ext}_{A_0}^{s-\alpha_s, t-\alpha_t}(M \otimes A//A_1, Z_2)$$

$$\oplus \sum_{\alpha \in B(d)} \operatorname{Ext}_A^{s-\alpha_s, t-\alpha_t}(M \otimes A/A(Sq^3, Sq^1), Z_2).$$

COROLLARY 4. *If* Q *is an* A *module which is free over* A_1, *the subalgebra generated by* Sq^1 *and* Sq^2, *then* $\operatorname{Ext}_A^{s,t}(Q, Z_2) = 0$ *for* $5s \geqq t-s+\epsilon$.

THEOREM 5. *Let* X *be a space in the stable category so that* $\Sigma X = RP^2$. *If* $E_r(X)$ *is the Adams spectral sequence converging to* $\pi_*^S(X)$, *then then* $E_5^{s,t} = E_\infty^{s,t}(X) = 0$ *for* $5s \geqq t-s+\epsilon$ *unless*

$$\begin{aligned} s &= 4k, & t-s &= 8k, \quad 8k+1, \quad 8k+2, \\ &= 4k+1, & t-s &= 8k+1, \quad 8k+2, \quad 8k+3, \\ &= 4k+2, & t-s &= 8k+2, \quad 8k+3, \quad 8k+7, \\ &= 4k+3, & t-s &= 8k+4, \quad 8k+8, \quad 8k+9, \end{aligned}$$

in which cases the groups are Z_2.

These elements represent the generators of the image of J and μ_j [1] on the bottom cell and the elements of order two in the im J and μ_j coextended on the top cell.

THEOREM 6. *There is a space* Im J *and a map* $f: S^0 \to \text{Im } J$ *so that* f_* *maps the image of* J *and the* μ*'s monomorphically onto the homotopy of* Im J.

In [1] a map $f: \Sigma^8 X \to X$ which represents an extension of a coextension of 8σ is studied. There it is proved that all iterations of f are essential.

THEOREM 7. *If* $\alpha: S^k \to X$ *is a stable map then*

$$S^{k+8j} \xrightarrow[\Sigma^{8j}\alpha]{} \Sigma^{8j} X \xrightarrow[f^j]{} X$$

is inessential for some j unless α is in one of the classes given by Theorem 5.

Some comments on the proof. Let the spectrum *bo* be the connected *BO* spectrum. Then we construct a Novikov resolution of S as follows

$$\begin{array}{c} \vdots \\ S_\sigma \to S_\sigma \wedge bo \\ \vdots \\ S_1 \to S_1 \wedge bo \\ S \to S \wedge bo. \end{array}$$

We apply the E_2 of the Adams spectral sequence to this tower and get a spectral sequence which converges to $H^{**}(A)$ except for $s=t$. If we consider the resolution $X \wedge S_\sigma$, where X is defined in Theorem 5, we can make an explicit calculation. Let

$$E_1^{s,t\sigma} = \mathrm{Ext}_A^{s-\sigma,t-\sigma}(\tilde{H}^*(X \wedge S_\sigma \wedge bo), Z_2).$$

PROPOSITION 8. $E_2^{s,t,\sigma} = \sum_{\alpha \in P^\sigma} \mathrm{Ext}_A^{s-\alpha_t,t-\alpha_t}(M \otimes A//A_1, Z_2)$ *for* $s>\sigma$ *where* P^σ *is the set of* σ*-degree polynomials in the polynomial algebra introduced above.*

PROPOSITION 9. $E_3^{s,t,\sigma} = E_\infty^{s,t,\sigma}$ *for* $s>\sigma$ *and thus is given by Theorem* 3.

Note that Proposition 9 alone gives an edge of $3\sigma > t-2$. The sharpened version of Theorem 3 follows from Proposition 9 and a closer analysis of the nature of $\mathrm{Ext}_A^{s,t}(M, Z_2)$.

The most direct route from Proposition 8 to the main theorem requires a geometric realization of the E_2 term of the above spectral sequence for S. Using this resolution and the homotopy functor we get a spectral sequence whose $E_2^{\sigma,t}$ term has an edge of $5\sigma \geqq t-\sigma+\epsilon$. The image of J has filtration 1. From this information the order of im J should follow directly but no direct route has been found. Hence to complete the argument, consider the space Y which is the fiber of the map $S \to K(Z, 0)$, and consider the resolution of Y given by $\cdots \to Y \wedge S_\sigma \to Y \wedge S_{\sigma-1} \to \cdots$. Calculation of the sort given in the proof of III 7.3 of [4] and applied to elements of filtration zero and one give a proof of Theorems 1 and 2.

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NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201