1. Introduction. In this note we state some results concerning the structure of a function which is the Fatou limit, on the distinguished boundary, of a bounded holomorphic function in a product of half-planes. Such a function is, equivalently, an element of $L^\infty(R^n)$ whose (distributional) Fourier transform is supported in the “first quadrant”, i.e. the set

$$Q^+ = \{ x_1 \geq 0, \ldots, x_n \geq 0 \}.$$ 

A typical result (consequence of Theorem 1) is that if such a function tends to a limit $\lambda$ as $x \to x^0$ from inside an open cone with vertex at $x^0$, then it tends “on the average” to $\lambda$ as $x \to x^0$ from inside any open cone with vertex at $x^0$. In particular, if such a function tends to limits in each of two open cones with a common vertex, these limits must be equal; for $n=1$ (when the cones are half-lines), this is a classical theorem of Pringsheim and Lindelöf. Theorem 2 is a Tauberian theorem for $H^\infty$ boundary values; specialized to one variable it yields (among other things) a new proof of Lindelöf’s theorem that a bounded analytic function which tends to a limit radially does so also within an angle, as well as an apparently new relation between the average behaviour of an $H^\infty$ boundary function to the left, and that to the right, of a given point. Theorem 3 shows that a much stronger localization of uniform convergence for, say, Fejér means, is valid for $H^\infty$ boundary functions than for bounded measurable functions generally; for example, if the restriction of an $H^\infty$ boundary function to a closed ball in $R^n$ is continuous, the Fejér means converge to it uniformly on the closed ball, not merely on subballs of smaller radius.

2. Notation and preliminary results. Let $P$ denote the open upper half-plane in the complex $z$-plane, and $R$ (the real axis) its boundary. By $P^n$ we denote the Cartesian product of $n$ copies of $P$, and $R^n$ is then the distinguished boundary of $P^n$. We use usual vector nota-
tions, denoting by \( z = (z_1, \ldots, z_n) \) a point of \( P^n \), \( z_j = x_j + iy_j \), and by \( x = (x_1, \ldots, x_n) \) a point of \( R^n \); \( dx \) denotes Lebesgue measure in \( R^n \), and \( a, b \) positive scalars.

By \( H^\infty(P^n) \) we denote the set of bounded holomorphic functions in \( P^n \). To each \( f \in H^\infty(P^n) \) is uniquely associated an element \( \phi \) of \( L^\infty(R^n) \), its boundary function, by the Fatou relation

\[
\lim_{a \to 0} f(x_1 + ia, \ldots, x_n + ia) = \phi(x_1, \ldots, x_n) \quad \text{a.e.}
\]

We can recover \( f \) from \( \phi \) by the Poisson formula

\[
(2.1) \quad f(z) = \int \phi(x - \xi) \hat{\rho}(\xi, y) d\xi
\]

(here and elsewhere the integration is over \( R^n \) if not specified otherwise) where

\[
(2.2) \quad \hat{\rho}(x, y) = \pi^{-n} \prod_{j=1}^{n} y_j(x_j^2 + y_j^2)^{-1}.
\]

By \( H^\infty(R^n) \) we denote the set of boundary functions of elements of \( H^\infty(P^n) \).

For \( h \in L^1(R^n) \), we write \( h_{(a)}(x) = a^{-n} h(a^{-1}x) \). Finally, \( B(x, a) \) denotes the closed ball in \( R^n \) with center \( x \) and radius \( a \).

3. Main results.

**Theorem 1.** Let \( \phi \in H^\infty(R^n) \), \( x^0 \in R^n \), and suppose there exist \( \lambda \in C, b > 0 \), and a function \( \sigma(a) \) tending to zero as \( a \to 0 \), with the following properties. For each \( a \leq 1 \), \( B(x^0, a) \) contains a ball \( K_a \) of radius \( ba \) such that \( \text{ess sup} |\phi(x) - \lambda|, x \in K_a \), does not exceed \( \sigma(a) \). Then, for every \( k \in L^1(R^n) \),

\[
(3.1) \quad (\phi \ast k_{(a)})(x^0) = \int \phi(x^0 - ax) k(x) dx \to \int k(x) dx
\]

as \( a \to 0 \).

**Remark 3.1.** Clearly, if \( \phi(x) \to \lambda \) as \( x \to x^0 \) within an open cone with vertex at \( x^0 \) the hypothesis is satisfied.

**Corollary 3.2.** Under the hypotheses of Theorem 1, if \( K \) is any open cone with vertex at \( x^0 \), the mean value of \( \phi \) over \( K \cap B(x^0, a) \) tends to \( \lambda \) as \( a \to 0 \).

**Proof.** Without loss of generality suppose \( x^0 \) is the origin. Choosing \( k \) so that \( k(-x) \) is the characteristic function of \( K \cap B(x^0, 1) \) gives the result.
Corollary 3.3. Under the hypotheses of Theorem 1, if \( \varphi \) is the boundary function of \( f \in H^\infty(P^n) \), then \( \lim_{a \to 0} f(x_0^0 + ia, \cdots, x_n^0 + ia) = \lambda \).

Proof. In view of (2.1) and (2.2),
\[
f(x_0^0 + ia, \cdots, x_n^0 + ia) = (\varphi * q(a))(x^0),
\]
where \( q(x) = \pi^{-1}(1 + x_1^2)^{-1} \cdots (1 + x_n^2)^{-1} \). Hence, choosing \( q \) for \( k \) in (3.1) gives the result.

Theorem 2. Let \( G \) denote a subset of \( L^1(\mathbb{R}^n) \) such that no nonnull \( \psi \in H^\infty(\mathbb{R}^n) \) satisfies

\[
\int \psi(ax)g(x)dx = 0, \quad \text{all } g \in G, \quad a > 0.
\]

If \( \varphi \in H^\infty(\mathbb{R}^n) \) satisfies
\[
\lim_{a \to 0} \int \varphi(ax)g(x)dx = \lambda \int g(x)dx, \quad \text{all } g \in G,
\]
then
\[
\lim_{a \to 0} \int \varphi(ax)h(x)dx = \lambda \int h(x)dx, \quad \text{all } h \in L^1(\mathbb{R}^n).
\]

Remark 3.4. We give three examples, for \( n = 1 \), of \( G \) for which the hypothesis is easily verified. In each example, \( G \) consists of a single function \( g \).
(i) \( g \) is the characteristic function of \([0, 1]\),
(ii) \( g \) is the characteristic function of \([-1, 1]\),
(iii) \( g(x) = (1 + x^2)^{-1} \).

Hence, if \( \varphi \in H^\infty(\mathbb{R}) \) satisfies

\[
\lim_{a \to 0} \int \varphi(ax)g(x)dx = \lambda \int g(x)dx
\]
for any one of these \( g \), it satisfies (3.3) for every \( g \in L^1(\mathbb{R}) \). In particular, the choice of \( g \) in (iii) yields

Corollary 3.5. If \( f \in H^\infty(P) \) satisfies \( \lim_{a \to 0} f(ia) = \lambda \), its boundary function \( \varphi \) satisfies (3.3) for every \( g \in L^1(\mathbb{R}) \).

From this it is quite easy to deduce that \( f(z) \to \lambda \) as \( z \to 0 \) in a Stolz angle (Lindelöf's theorem). Other specializations of \( g \) lead to these typical results.\(^1\)

\(^1\) (Cf. next page, line 2.)
COROLLARY 3.6. If $\varphi \in H^\omega(R)$ satisfies $\lim_{a \to 0} a^{-1} \int_0^a \varphi(x) \, dx = \lambda$, then $\lim_{a \to 0} a^{-1} \int_0^a \varphi(x) \, dx = \lambda$.

COROLLARY 3.7. If $\varphi \in H^\omega(R)$ satisfies $\lim_{a \to 0} (2a)^{-1} \int_{-a}^a \varphi(x) \, dx = \lambda$, then $\lim_{a \to 0} a^{-1} \int_0^a \varphi(x) \, dx = \lambda$.

The last three corollaries can also be proved quite directly by means of Wiener's Tauberian theorem; our proofs are however simpler, and independent of Wiener's theorem.

DEFINITION. A compact set $E \subset \mathbb{R}^n$ is well-rounded if constants $b, \delta$ can be found such that for every $x \in E$ and $a \leq b$, $E \cap B(x, a)$ contains a ball of radius $\delta a$.

EXAMPLES. A closed interval in $\mathbb{R}$ is well-rounded (we can take $\delta = \frac{1}{2}$); a polygonal region in $\mathbb{R}^2$, or any region with piecewise smooth boundary and no cusps, is well-rounded.

THEOREM 3. Let $E$ be a well-rounded compact set in $\mathbb{R}^n$. Suppose $\varphi$ is in $H^\omega(\mathbb{R}^n)$ and coincides a.e. on $E$ with a continuous function $\varphi_0$. Then, if $k$ is any integrable function on $\mathbb{R}^n$ satisfying $\int k \, dx = 1$, $(\varphi * k)(x)$ converges to $\varphi_0(x)$, uniformly for $x \in E$, as $a \to 0$.

REMARK 3.8. If $F$ is bounded and holomorphic in the unit polydisc $U^n$ (see Rudin [2]), and $\Phi$ its boundary function on the torus $T^n$, then $f(z_1, \ldots, z_n) = F(\exp[i z_1], \ldots, \exp[i z_n])$ is in $H^\omega(P^n)$ and its boundary function is $\Phi(\exp[i z_1], \ldots, \exp[i z_n])$. This enables us to apply the results of this paper to the polydisc-torus framework. In particular, Theorem 3 implies that quite general summability methods, applied to the Fourier series of $\Phi$, converge uniformly on well-rounded subsets of $T^n$ where $\Phi$ is continuous ("well-rounded" being suitably redefined for the torus). For example, taking $n = 1$, $k(x) = \pi^{-1} \sin x / x$, we deduce that the Fejér means of the partial sums of the Fourier series of $\Phi \in H^\omega(T)$ converge uniformly to $\Phi$ on each closed interval to which the restriction of $\Phi$ is continuous.

4. Method of proof. Full details shall be given elsewhere; here we merely indicate the main ideas. The proofs are by "soft analysis", based upon two crucial properties of $H^\omega(\mathbb{R}^n)$:

(i) $H^\omega(\mathbb{R}^n)$ is a weak* closed subspace of $L^\omega(\mathbb{R}^n)$.

(ii) Quasi-analyticity: a function in $H^\omega(\mathbb{R}^n)$ which vanishes on a ball (or even on a set of positive measure) vanishes identically.

Let us write $\langle \varphi, g \rangle$ to denote $\int \varphi(x) g(-x) \, dx$, and define $^0H^\omega(\mathbb{R}^n) = \{ g \in L^1(\mathbb{R}^n) : \langle \varphi, g \rangle = 0 \text{ for all } \varphi \in H^\omega(\mathbb{R}^n) \}$.

LEMMA 4.1. Given $h \in L^1(\mathbb{R}^n)$, $\epsilon > 0, b < 1$, there is a constant $C(h, \epsilon, b)$ such that if $B$ is any ball of radius $b$ lying in the unit ball of $\mathbb{R}^n$, there
exists $g \in \mathcal{O}H^\alpha(R^n)$ such that $\|g\|_{1} \leq C(h, \epsilon, b)$ and

$$\int_{R^n \setminus B} |h(x) - g(x)| \, dx < \epsilon. \quad (4.1)$$

**Proof (Outline).** For each $B$, the existence of $g \in \mathcal{O}H^\alpha(R^n)$ satisfying (4.1) follows by a standard duality argument from (i) and (ii). A compactness argument allows an estimate for $\|g\|_{1}$ independent of the particular choice of $B$.

**Proof of Theorem 1.** Without loss of generality, take $x^0 = \text{origin}$, $\lambda = 0$. We have

$$\int \varphi(ax)k(-x)dx = \int \varphi(ax)(k(-x) - g(x))dx$$

where $g = g_a$ is an element of $\mathcal{O}H^\alpha(R^n)$ that is at our disposal (here, in a nutshell, is the essence of our method).

By hypothesis, $B(0, a)$ contains a ball $K_a$ of radius $ba$ on which $\text{ess sup} |\varphi(x)| \leq \sigma(a)$, hence on the ball $J_a = a^{-1}K_a$, which is contained in the unit ball and has radius $b$, $\text{ess sup} |\varphi(ax)| \leq \sigma(a)$. Now apply the lemma, with $h(x) = k(-x)$; we get, for a suitable $g = g_a \in \mathcal{O}H^\alpha(R^n)$

$$\left| \int \varphi(ax)k(-x)dx \right| = \left| \int_{J_a} \varphi(ax)[k(-x) - g(x)]dx \right| + \int_{R^n \setminus J_a} \varphi(ax)[k(-x) - g(x)]dx$$

$$\leq \sigma(a)(\|k\|_{1} + C(k, \epsilon, b)) + \|\varphi\|_{\infty \cdot \epsilon},$$

hence

$$\limsup_{a \to 0} \left| \int \varphi(ax)k(-x)dx \right| \leq \|\varphi\|_{\infty \cdot \epsilon},$$

and, $\epsilon$ being arbitrary, the lim sup is zero.

**Proof of Theorem 2.** We may suppose $\lambda = 0$. Define

$$M = \left\{ h \in L^1(R^n) : \lim_{a \to 0} \int \varphi(ax)h(x)dx = 0 \right\}.$$ 

$M$ is a closed subspace of $L^1(R^n)$ containing all dilates of $g$ (i.e. the functions $x \to g(bx)$, $b > 0$) for each $g \in G$. Also, $\mathcal{O}H^\alpha(R^n) \subset M$. Hence the theorem (i.e. $M = L^1(R^n)$) will be proved if $\mathcal{O}H^\alpha(R^n)$ together with the dilates of functions in $G$ span $L^1(R^n)$. By a duality argument, based on weak$^*$ closure of $H^\alpha(R^n)$, this is ensured by the nonexistence of a nonnull solution $\psi \in \mathcal{O}H^\alpha(R^n)$ to (3.2).
The proof of Theorem 3 is along similar lines to that of Theorem 1, and shall not be given here.

5. Concluding remarks.

(a) It is natural to ask whether in Theorem 1 $K_n$ may be required merely to be a *set* of measure $c a^n$, rather than a *ball* of this measure. To prove this by our method would require a corresponding strengthening of Lemma 4.1 whereby $B$ is allowed to be any set of measure $c$ lying in the unit ball. This we have only been able to carry out for $n = 1$; for $n = 1$ we can also prove an analogous extension of Theorem 3, in terms of a weaker definition of "well-rounded" based on density.

(b) In the theorems of this paper, $H^n(R^n)$ can be replaced by $S(\Lambda)$, where $\Lambda$ is a closed subset of (the dual) $R^n$, defined as the set of $\varphi \in L^n(R^n)$ whose (distributional) Fourier transform is supported in $\Lambda$, provided that $S(\Lambda)$ has the quasi-analytic property.

(c) We do not know in general how to decide whether or not a given $G \subseteq L^1(R^n)$ satisfies the hypothesis of Theorem 2. We wish also to call the readers' attention to the analogous problem, for which $G \subseteq L^1(R^n)$ is there no nonnull solution $\psi$ in $L^n(R^n)$ of (3.2)? This problem, the analog for dilates of the famous Wiener problem for translates arises naturally in studying boundary behaviour of harmonic functions. The case $n = 1$ is reducible to the Wiener problem (cf. Gehring [1, pp. 107–110] for something similar), but the case $n > 1$ seems to us fundamentally different in nature.

References