THE P-SINGULAR POINT OF THE P-COMPACTIFICATION FOR \( \Delta u = Pu \)

BY Y. K. Kwon AND L. Sario

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ABSTRACT. By means of the \( P \)-algebra \( M_p(R) \) of bounded energy-finite Tonelli functions on a Riemannian manifold \( R \), we construct the \( P \)-compactification \( R_p \) of \( R \) as a quotient space of the Royden compactification. The \( P \)-singular point \( s_p \) is explicitly characterized in terms of the density \( P \). The dimension of the space \( \text{PBE}(R) \) of bounded energy-finite \( P \)-harmonic functions on \( R \) is shown to exceed exactly by one the cardinality of the \( P \)-harmonic boundary \( \Delta_P \) if \( s_p \not\in \Delta_P \). If \( s_p \in \Delta_P \) one can replace the density \( P \) by another \( Q \) such that \( \text{dim} \ Q \text{BE}(R) = \text{dim} \ P \text{BE}(R) \) and a \( Q \)-singular point does not exist.

In the study of the equation \( \Delta u = Pu \), \( P \geq 0 \), on a Riemannian manifold \( R \), it is useful to consider the algebra \( M_p(R) \) of bounded energy-finite Tonelli functions. With \( M_p(R) \) one associates the \( P \)-compactification \( R^*_p \) of \( R \) on which every \( f \in M_p(R) \) has a continuous extension (Nakai-Sario [4]). An interesting phenomenon is the occurrence of the \( P \)-singular point \( s \in R^*_p \) defined by \( f(s) = 0 \) for every \( f \in M_p(R) \).

In the present note we construct \( R^*_p \) as a quotient space of the Royden compactification \( R^* \). Necessary and sufficient for the existence of an \( s \) is that \( 1 \not\in M_p(R) \). If an \( s \) exists, it is unique. We shall give an explicit characterization of \( s \) in terms of \( P \), thus establishing a link with a property considered by Glasner and Katz [2].

We then show that if \( s \) lies on the \( P \)-harmonic boundary \( \Delta_P \), the cardinality of \( \Delta_P \) exceeds exactly by one the dimension of the space of bounded energy-finite \( P \)-harmonic functions on \( R \).

If \( s \) does not lie on \( \Delta_P \), it is removable in the sense that there exists a density \( Q \) on \( R \) without a \( Q \)-singular point such that \( \text{dim} \ Q \text{BE}(R) = \text{dim} \ P \text{BE}(R) = \text{the cardinality of} \Delta_P \).

1. On a smooth Riemannian \( n \)-manifold \( R \), \( n \geq 2 \), consider \( P \)-
harmonic functions, i.e. solutions of the elliptic partial differential equation

$$\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{g} g^{ij} \frac{\partial u}{\partial x^j} \right) = Pu.$$  

Here $x = (x^1, \cdots, x^n)$ is a local coordinate, $(g^{ij})$ the inverse of the matrix $(g_{ij})$ of the fundamental metric tensor of $\mathbb{R}$, $g$ the determinant of $(g_{ij})$, and $P (\neq 0)$ a nonnegative continuous function on $\mathbb{R}$.

Denote by $M_P(\mathbb{R})$ the algebra of bounded Tonelli functions $f$ on $\mathbb{R}$ with finite energy integrals $E_R(f) = E_R(f, f)$. Here the inner product $E_R(f, g)$ is defined by

$$E_R(f, g) = \int_{\mathbb{R}} \left[ \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} + Pf \right] dV,$$

with $dV$ the volume element. $\star 1$

Let $f \in M_P(\mathbb{R})$. Given a regular subregion $\Omega$ of $\mathbb{R}$, construct the function $u$ on $\mathbb{R}$ such that $u = f$ on $\mathbb{R} - \Omega$ and $\Delta u = Pu$ on $\Omega$. The energy principle (Royden [5]) reads

$$E_B(u) \leq E_R(f), \quad u \in M_P(\mathbb{R}).$$

If $g \in M_P(\mathbb{R})$ and $g \equiv 0$ on $\mathbb{R} - \Omega$, then $E_R(g, u) = 0$.

2. Denote by $M(\mathbb{R})$ the Royden algebra and by $R^*$ the Royden compactification of $\mathbb{R}$ (cf. e.g. Chang-Sario [1] and Sario-Nakai [6]). In view of $M_P(\mathbb{R}) \subseteq M(\mathbb{R})$ every function $f \in M_P(\mathbb{R})$ has a continuous extension to $R^*$.

2. For $x, y \in R^*$ set $x \sim y$ if $f(x) = f(y)$ for all $f \in M_P(\mathbb{R})$. Clearly "$\sim$" is an equivalence relation. Denote by $R^*_p$ the quotient space $R^*/\sim$. Let $\pi_P : R^* \to R^*_p$ be the natural projection.

**Proposition 1.** The space $R^*_p$ endowed with the quotient topology is a compact Hausdorff space and contains $\mathbb{R}$ as a connected open dense subset.

**Proposition 2.** Every function in $M_P(\mathbb{R})$ has a continuous extension to $R^*_p$, and $M_P(\mathbb{R})$ separates points in $R^*_p$.

We shall call $R^*_p$ the $P$-compactification and $M^*_p(\mathbb{R})$ the $P$-algebra of $\mathbb{R}$. For the continuations of $f \in M_P(\mathbb{R})$ to $R^*$ and $R^*_p$ we use the same notation $f$.

$P$-regularity can be given the following explicit characterization:

3. A point $x \in R^*_p$ will be called $P$-regular or $P$-singular according as there does or does not exist a function $f \in M_P(\mathbb{R})$ with $f(x) \neq 0$. By
Theorem 1. A point \( x \in \mathbb{R}^n \) is \( P \)-regular if and only if the density function \( P \) has a finite integral at \( x \), i.e. there exists an open neighborhood \( U \) of \( x \) in \( \mathbb{R}^n \) with \( \int_{U \cap \mathbb{R}} P \, dV < \infty \).

Proof. If \( x \) is \( P \)-regular, there exists a function \( f \in M_p(\mathbb{R}) \) with \( f(x) \neq 0 \). Choose \( \epsilon > 0 \) such that \( |f(x)| > \epsilon \). Then \( U = \{ y \in \mathbb{R}^n \mid |f(y)| > \epsilon \} \) is an open neighborhood of \( x \) in \( \mathbb{R}^n \). Since

\[
\int_{U \cap \mathbb{R}} P \, dV \leq \frac{1}{\epsilon^2} \int_{U \cap \mathbb{R}} P^2 \, dV \leq \frac{1}{\epsilon^2} E_{\mathbb{R}}(f),
\]

\( P \) has a finite integral at \( x \).

Conversely suppose that there exists an open neighborhood \( U \) of \( x \) in \( \mathbb{R}^n \) with \( \int_{U \cap \mathbb{R}} P \, dV < \infty \). Since \( \mathbb{R}^n - \pi_1^{-1}(U) \) and \( \pi_1^{-1}(x) \) are disjoint closed sets in \( \mathbb{R}^n \), we can choose a function \( g \in M(\mathbb{R}) \) such that \( 0 \leq g \leq 1 \), \( g \mid \pi_1^{-1}(x) = 1 \), and \( g \mid \mathbb{R}^n - \pi_1^{-1}(U) = 0 \). Then we have

\[
\int_{\mathbb{R}} P g^2 \, dV = \int_{\mathbb{R} \setminus \pi_1^{-1}(U)} P g^2 \, dV + \int_{\pi_1^{-1}(U)} P g^2 \, dV \leq \int_{\mathbb{R} \setminus \pi_1^{-1}(U)} P \, dV + \int_{U \cap \mathbb{R}} P \, dV < \infty.
\]

Thus \( g \in M_p(\mathbb{R}) \) and \( g(x) = 1 \), i.e. \( x \) is \( P \)-regular.

A point \( s \in \mathbb{R}^n \) is \( P \)-singular if and only if \( \int_{U \cap \mathbb{R}} P \, dV = \infty \) for each open neighborhood \( U \) of \( s \) in \( \mathbb{R}^n \).

Remark. If there exist no \( P \)-singular points, then we have the special case \( \mathbb{R}^n = \mathbb{R}^n \) studied in Royden [5]. In our note we assume that \( s \) exists. The concept of a \( P \)-singular point was introduced in Nakai-Sario [4], and the term "\( P \) has a finite integral at \( x \)" in Glasner-Katz [2].

4. We write \( f = \text{BE-lim}_n f_n \) on \( \mathbb{R} \) if the sequence \( \{ f_n \} \) is uniformly bounded on \( \mathbb{R} \), converges to \( f \) uniformly on compact subsets of \( \mathbb{R} \), and \( E_{\mathbb{R}}(f_n - f) \to 0 \) as \( n \to \infty \). In view of the BD-completeness of Royden’s algebra \( M(\mathbb{R}) \) (e.g. Sario-Nakai [6]) it is not difficult to see that the \( P \)-algebra \( M_p(\mathbb{R}) \) is BE-complete.

Let \( \Delta_p = \pi_p(\Delta) \) and denote by \( M_{p0}(\mathbb{R}) \) the space of functions in \( M_p(\mathbb{R}) \) with compact supports in \( \mathbb{R} \), and by \( M_{p\Delta}(\mathbb{R}) \) the space of BE-limits in \( M_p(\mathbb{R}) \) of functions in \( M_{p0}(\mathbb{R}) \). As in the case of the potential subalgebra \( M_\Delta(\mathbb{R}) \) (cf. [3]) we have the duality:

Proposition 3. \( M_{p\Delta}(\mathbb{R}) = \{ f \in M_p(\mathbb{R}) \mid f = 0 \text{ on } \Delta_p \} \).

Proof. It suffices to show that
THE P-COMPACTIFICATION FOR \( \Delta u = Pu \)

\[
M_{P\Delta}(R) = \{ f \in M_P(R) \mid f \equiv 0 \text{ on } \Delta \}.
\]

Since \( M_{P\Delta}(R) \subset M_\Delta(R) \), \( M_{P\Delta}(R) \subset \{ f \in M_P(R) \mid f \equiv 0 \text{ on } \Delta \} \) (cf. [3]). Conversely, suppose that \( f \in M_P(R) \) vanishes identically on \( \Delta \). Since \( M_P(R) \) is a lattice, we may assume that \( f \geq 0 \). Choose a sequence \( \{ f_n \} \) of functions in \( M(R) \) with compact supports in \( R \) such that \( 0 \leq f_n \leq f \) and \( f = \text{BD-lim}_n f_n \) on \( R \). By Lebesgue’s dominated convergence theorem

\[
\int_R Pf^2 \, dV = \lim_{n \to \infty} \int_R P f_n^2 \, dV.
\]

Consequently \( f \in M_{P\Delta}(R) \) as desired.

**Corollary.** \( M_{P\Delta}(R) \) is an ideal of \( M_P(R) \).

5. We turn to the vector space \( \text{PBE}(R) \) of bounded energy-finite \( P \)-harmonic functions on \( R \).

We maintain (for Royden’s compactification cf. Glasner-Katz [2]):

**Theorem 2.** The vector space \( \text{PBE}(R) \) is \( m \)-dimensional if and only if the \( P \)-harmonic boundary \( \Delta_P \) consists of \( m+1 \) points whenever \( s \in \Delta_P \). If \( s \) does not lie on \( \Delta_P \), then \( \dim \, \text{PBE}(R) \) equals the cardinality of \( \Delta_P \).

For the proof we first establish the orthogonal decomposition:

**Lemma 1.** \( M_P(R) = \text{PBE}(R) \oplus M_{P\Delta}(R) \).

**Proof.** Let \( f \in M_P(R) \). Since \( M_P(R) \) is a vector lattice we may assume that \( f \geq 0 \) on \( R \).

For a regular exhaustion \( \{ R_n \} \) of \( R \) consider the functions \( u_n \in M_P(R) \) such that \( u_n \in \text{PBE}(R_n) \) and \( u_n \equiv f \) on \( R - R_n \). By the energy principle (cf. 1),

\[
E_R(u_n) \leq E_R(f) < \infty,
\]

\[
E_R(u_n) = E_R(u_{n+p}) + E_R(u_{n+p} - u_n)
\]

for all \( n, p \geq 1 \). Hence \( \{ u_n \} \) is \( E \)-Cauchy. Since it is uniformly bounded on \( R \), we may assume that it converges to a \( P \)-harmonic function, uniformly on compact subsets of \( R \) (cf. Royden [5]).

Set \( u = \text{BE-lim}_n u_n \) and \( g = \text{BE-lim}_n (f - u_n) \) on \( R \). Then \( f = u + g \) is the desired decomposition. Its uniqueness is obvious by the definition of \( M_{P\Delta}(R) \).

**Lemma 2.** \( R \in O_{\text{PBE}} - O_G \) if and only if \( \Delta_P = \{ s \} \).
Proof. If $\Delta_P = \{s\}$, $M_P(R) = M_{PA}(R)$ and $PBE(R) = \{0\}$.

Conversely, suppose that there exists a $P$-regular point $x$ in $\Delta_P$. Choose open neighborhoods $U, V$ of $s$ in $R^n_s$ such that $x \in U$ and $V \subset U$. Since $\pi^{-1}_P(V)$ and $\pi^{-1}_P(R^n_s - U)$ are disjoint closed sets in $R^n$, we can construct an $f \in M_P(R)$ with $0 \leq f \leq 1$, $f|\pi^{-1}_P(V) = 0$, and $f|\pi^{-1}_P(R^n_s - U) = 1$.

Let $f = u + g$ be the decomposition in Lemma 1. Then $u$ is a non-constant $PBE$-function and therefore $R \in O_{PBE} - O_P$.

Proof of Theorem 2. Let $\{x_1, \ldots, x_m\}$ be a finite subset of $\Delta_P - s$. As in the proof of Lemma 2, we can construct nonconstant functions $u_i$ in $PBE(R)$ with $u_i(x_i) = \delta_{ij}$. Since the $u_i$ are linearly independent, $\dim PBE(R) = \infty$ whenever $\Delta_P$ is an infinite set.

Suppose that the cardinality of $\Delta_P$ is $m + 1$ and that $s \in \Delta_P$. For any $u \in PBE(R)$, $u - \sum_{i=1}^m u(x_i)u_i \in PBE(R) \cap M_{PA}(R) = \{0\}$ and we conclude that $\dim PBE(R) = m$ is the cardinality of $\Delta_P - s$.

The proof in the case in which the cardinality of $\Delta_P$ is finite and $s \in \Delta_P$ is the same.

6. We have seen that the dimension of the space $PBE(R)$ is equal to the cardinality of the $P$-harmonic boundary whenever the $P$-singular point $s$ does not lie on $\Delta_P$. Thus the existence of $s$ in this case is, in a sense, of little significance as far as the relation of $PBE(R)$ and $\Delta_P$ is concerned. It is natural to ask: Can one replace the density $P$ by another, $Q$, such that $\dim PBE(R) = \dim QBE(R)$, and a $\Delta_P$-singular point does not exist?

First we prove:

Theorem 3. The $P$-singular point $s$ lies on $R^n_s - \Delta_P$ if and only if there exists a $PBE$-function $u$ on $R$ such that $u = 1$ on $\Delta_P$.

Proof. The necessity is trivial since $PBE(R) \subset M_P(R)$. For the sufficiency choose an $f_s \in M_P(R)$ for a given $x \in \Delta_P$ such that $f_s \geq 0$ and $f_s(x) > 0$. Since $\Delta_P$ is compact we can construct a function $f \in M_P(R)$ with $f \geq 0$ and $f|\Delta_P > 0$. Set $\alpha = \min \Delta_P f > 0$, and let $\alpha^{-1}(f|\Delta_P) = u + g$ be the decomposition in Lemma 1. Then $u$ has the required property.

Theorem 4. If $P, Q$ are densities on $R$ which coincide on an open neighborhood $U$ of $\Delta$ in $R^*$, then $\dim PBE(R) = \dim QBE(R)$.

Proof. First we show that $\Delta_P$ and $\Delta_Q$ have the same cardinality. Let $\pi_P : R^* \rightarrow R^n_s$ be the natural projection and let $\pi_P(x) \neq \pi_P(y)$ for $x, y \in \Delta$. Then there exists a function $f \in M_P(R)$ with $f(x) \neq f(y)$. Choose an open neighborhood $V$ of $\Delta$ in $R^*$ such that $\bar{V} \subset U$ and a
function \( g \in M(R) \) such that \( 0 \leq g \leq 1 \), \( g|V = 1 \), and \( g|U^* = U = 0 \). Clearly \( fg \in M_Q(R) \) and \( (fg)(x) \neq (fg)(y) \), i.e. \( \pi_Q(x) \neq \pi_Q(y) \). We infer that the cardinalities of \( \Delta_P \) and \( \Delta_Q \) coincide, and therefore \( \dim PBE(R) = \infty \) if and only if \( \dim QBE(R) = \infty \).

Let the common cardinality of \( \Delta_P \) and \( \Delta_Q \) be \( k < \infty \). If the \( P \)-singular point \( s_P \) belongs to \( \Delta_P \), choose \( x \in \Delta \) such that \( \pi_P(x) = s_P \). Then it is easily seen that \( \pi_Q(x) \) is the \( Q \)-singular point and \( \pi_Q(x) \in \Delta_Q \). By Theorem 2 it follows that \( \dim PBE(R) = \dim QBE(R) = k - 1 \) (resp. \( k \)) if \( s_p \in \Delta_P \) (resp. \( s_p \in \Delta_P \)).

If a \( P \)-singular point \( s_P \) exists but does not lie on \( \Delta_P \), then it may be called a “removable” \( P \)-singular point in the following sense:

**THEOREM 5.** If the \( P \)-singular point \( s_P \) lies on \( R_p^* - \Delta_P \), there exists a density \( Q \) on \( R \) such that \( \dim QBE(R) = \dim PBE(R) \) and \( \int_R Q \, dV < \infty \).

**PROOF.** Choose open neighborhoods \( U, V \) of \( s_P \) in \( R^* \) such that \( V \subseteq U \) and \( \overline{U} \cap \Delta_P = \emptyset \). Since \( \pi_P^{-1}(V) \) and \( R^* - \pi_P^{-1}(U) \) are disjoint closed subsets of \( R^* \) there exists a function \( f \in M_P(R) \) with \( 0 \leq f \leq 1 \), \( f|\pi_P^{-1}(V) = 0 \), and \( f|R^* - \pi_P^{-1}(U) = 1 \).

Set \( Q = f^*P \). Then \( \int_R Q \, dV = \int_R Pf^* dV \leq E_R(f) < \infty \), and by Theorem 4 we have \( \dim QBE(R) = \dim PBE(R) \).

**BIBLIOGRAPHY**