THE ADAMS-NOVIKOV SPECTRAL SEQUENCE FOR THE SPHERES

BY RAPHAEL ZAHLER¹

Communicated by P. E. Thomas, June 17, 1970

The Adams spectral sequence has been an important tool in research on the stable homotopy of the spheres. In this note we outline new information about a variant of the Adams sequence which was introduced by Novikov [7]. We develop simplified techniques of computation which allow us to discover vanishing lines and periodicity near the edge of the \( E_2 \) -term, interesting elements in \( E_2^{*,*} \), and a counterexample to one of Novikov's conjectures. In this way we obtain independently the values of many low-dimensional stems up to group extension. The new methods stem from a deeper understanding of the Brown-Peterson cohomology theory, due largely to Quillen [8]; see also [4]. Details will appear elsewhere; or see [11].

When \( p \) is odd, the \( p \)-primary part of the Novikov sequence behaves nicely in comparison with the ordinary Adams sequence. Computing the \( E_2 \)-term seems to be as easy, and the Novikov sequence has many fewer nonzero differentials (in stems \( \leq 45 \), at least, if \( p = 3 \)), and periodicity near the edge. The case \( p = 2 \) is sharply different. Computing \( E_2 \) is more difficult. There are also hordes of nonzero differentials \( d_n \), but they form a regular pattern, and no nonzero differentials outside the pattern have been found. Thus the diagram of \( E_4 \) \((=E_\infty \) in dimensions \( \leq 17 \)) suggests a vanishing line for \( E_\infty \) much lower than that of \( E_4 \) of the classical Adams spectral sequence [3].

It is a pleasure to thank Arunas Liulevicius, my thesis adviser, for his help. In particular, parts of the proofs of Proposition 1 and Theorem 7 are due to him. I am also grateful to many others for their suggestions, and especially to Frank Adams.

1. The spectral sequence. The construction of the classical Adams spectral sequence for the spheres [1] works equally well if the spec-

¹ Most of the results announced in this paper were in the author's doctoral dissertation, written under an NSF graduate fellowship at the University of Chicago.
trum $K(Z_p)$ representing ordinary cohomology is replaced by an arbitrary ring spectrum $X$. If $X$ satisfies certain conditions, the $E_2$-term of the resulting sequence will be isomorphic to

$$\text{Ext}_A^*(\Delta^X, \Lambda^X),$$

where $A^X = X^*(X)$ is the algebra of operations in $X$-cohomology theory and $\Delta^X = \pi_*(X)$ is the coefficient ring. Novikov showed [7] that if $X = MU$ (the spectrum representing complex cobordism) this multiplicative spectral sequence converges to the stable homotopy ring $\pi_*^S$:

$$E^s_{t+1} \cong F_*^{s-t-1}/F_*^{s-t-1},$$

where $F_*$ is a filtration of $\pi_*^S$. Furthermore, if $X' = BP_p$, the Brown-Peterson spectrum [4] for the prime $p$, the resulting spectral sequence $\{pE_r, pd_r\}$ is exactly the $p$-primary part $\{E_r \otimes Q_p, d_r \otimes Q_p\}$ of the $MU$ spectral sequence ($Q_p$ is the ring of rational numbers with denominators prime to $p$).

Not much is known about the $MU$ spectral sequence, because even limited computations of $E_2$ have been difficult. This is regrettable, since what is known indicates that the Novikov sequence has certain a priori advantages over the usual one. The nonzero terms are sparse, for example: $pE^*_n = 0$ if $t \equiv 0 \mod 2(p-1)$. Furthermore, almost all of the image of the $J$-homomorphism [2], [9] lies on the line $t = 1$, in the following sense. According to Novikov, $E^2_{2t} = Z_m(\langle \alpha_t \rangle)$, a cyclic group with generator $\alpha_t$, isomorphic to the image of $J$ in dimension $2t-1$ (isomorphic to $Z_2$ if $2t-1 = 5 \mod 8$). There is a map $q_1: \pi_*^S E^2_{n+1} \to E^2_{n+1}$ such that an element of $E^2_{n+1}$ survives to $E_\infty$ if it belongs to $\text{im } q_1$. Furthermore, if $q_1$ denotes the restriction of $q_1$ to $\text{im } J$, then [7, Chapters 10 and 11]

- (1) if $n = 8k + 1$, $E^2_{1,n+1} = E^2_{1,n+1} = Z_2$;
- (2) if $n = 8k + 3$ ($k > 0$), then $\text{im } q_1$ has index 2 in $E^2_{1,n+1} = Z_{m(4k+3)}$, and $q_1$ has kernel $Z_2$; in fact, $d_3 \alpha_{4k+3} = h^2 \alpha_{4k} \neq 0$;
- (3) if $n = 8k + 5$, $E^2_{2,n+1} = Z_{2}$ does not survive to $E_\infty$; in fact, $d_3 \alpha_{4k+3} = h^2 \alpha_{4k+1} \neq 0$;
- (4) if $n = 8k + 7$, $\text{im } q_1 = Z_{m(4k+4)} = E^2_{1,n+1} = E^2_{1,n+1}$.

Here $h = \alpha_1$.

2. Quillen's algebra. Novikov knew that, given a prime $p$, the algebra $A^{BP} = BP^*(BP)$ was much simpler than $A^{MU} \otimes Q_p$, but he did not have complete information about $A^{BP}$. Later, Quillen [8] discovered an idempotent $e$, which split the spectrum $MUQ_p$ into a sum of suspensions of the spectrum $BP_p$ [4]. Now
\[ \pi_* (BP) = Q_p[k_1, k_2, \ldots], \quad H_* (BP) = Q_p[m_1, m_2, \ldots], \]

with \(|k_i| = -|m_i| = -2(p^i - 1)|. We can take \(m_i = (1/p^i)he[CP^{p^i-1}]\); the Hurewicz homomorphism \(h\) is monic, and may be computed using Quillen’s formal-group techniques [11] or standard methods. Thanks to the idempotent \(e\), Quillen and Adams were able to write down explicit formulas for the Hopf-algebra structure of the algebra of operations \(A^BP( = A, \text{for short})\).

First, there is a coalgebra \(R\) of operations, free as a \(Q_p\)-module on generators \(r_E\), where \(E\) runs over all finitely nonzero sequences \((e_1, e_2, \cdots)\) of nonnegative integers and \(|r_E| = 2(\sum (p^i - 1)e_i)\). The diagonal map is given by \(\phi^* r_E = \sum_{E' + E'' = E} r_{E'} \otimes r_{E''}\). Then \(\Lambda' = \pi_* (BP)\) is an algebra over the coalgebra \(R\), with action given (via the Hurewicz map) by \(r_E m_n = m_{n-i}\) if \(e_i = p^{n-i}\) and all other \(e_j\) are zero, and \(r_E m_n = 0\) otherwise. Moreover, multiplication by an element \(\lambda\) of \(\Lambda'\) is also a \(BP\)-cohomology operation, and in fact every operation can be written as a (possibly infinite) sum \(\sum \lambda r_{E_i}\) in which the degree of each \(\lambda r_{E_i}\) is a constant independent of \(i\). Unfortunately, the composition \(r_{E'} r_{E''}\) of two operations in \(R\) does not usually lie in \(R\); however, it can be written uniquely as a finite sum \(r_{E'} r_{E''} = \sum K c_K r_K\) with \(c_K \in \Lambda'\), using the methods of [11] or those of [4]. This enables us to express compositions \((\lambda r_{E'})(\Lambda' r_{E''})\) in the form \(\sum \lambda r_{E_i}\). Thus the algebra \(A\) of all operations is the completed tensor product \(\Lambda' \hat{\otimes} R\).

**Proposition 1.** Let \(\Lambda\) be the two-sided ideal in \(A\) generated by all elements of \(\Lambda\) of negative degree. Let \(A_p/(Q_0)\) be the algebra of reduced Steenrod \(p\)th powers [6]. Then there is an isomorphism \(f:A/\Lambda \cong A_p/(Q_0)\).

**Proof.** Let \(Th:BP_p\to K(Z_p)\) be the \(Z_p\) Thom class. Then

\[ \bar{f} = Th_*: [BP, BP] \to [BP, K(Z_p)] \]

\[ \begin{array}{c|c}
\| & \| \\
A & H*(BP;Z_p) \\
\| & \| \\
A_p/(Q_0) & \\
\end{array} \]

satisfies

\[ \bar{f}(k^{E_p}) = c(0^p), \quad E = 0 \ [6]; \]

\[ = 0, \quad \text{otherwise}; \]

where \(c\) is the canonical antiautomorphism. The map \(\bar{f}\) induces the required \(f\) on \(A/\Lambda\).
A generator $r_R$ of $R$ is \textit{indecomposable} if it cannot be expressed as a finite sum $r_R = \sum \lambda_i r_i r'_i$, where $\lambda_i \in \Lambda'$; $R_i, R'_i \in R$; and $|R_i|, |R'_i| > 0$.

**Theorem 2.** The generator $r_R$ of $R$ is indecomposable if and only if $E = (p^i, 0, 0, \cdots), i \geq 0$. Moreover, $p_{r(p^i, 0, 0, \cdots)}$ is decomposable.

The proof is obtained by noticing certain pleasant properties of the multiplication table for $R$ and applying them in the proper sequence.

3. \textbf{Resolutions over $A$.} To compute Ext we must construct resolutions over $A$, which seems difficult at first glance since $R$ is not an algebra, $A$ is not connected, and the ground ring $Q_p$ is not a field. The next proposition shows how to circumvent some of these difficulties. Define the filtrations $F^i\Lambda' = \sum_{i \leq 2s} (\Lambda')^i$, $F^i A = F^i \Lambda' \hat{\otimes} A$, and $F^i M = (F^i A) M$ if $M$ is an $A$-module. We have

$$0 \rightarrow F^1 M \rightarrow M \rightarrow \operatorname{cok} i \rightarrow 0.$$ 

Write $JM$ for $\operatorname{cok} i$; then $J$ is easily made into a functor on the category of $A$-modules.

**Proposition 3.** \textit{There exist complexes}

$$C: \cdots \rightarrow C_i \xrightarrow{d_i} C_{i-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{d_1} C_0 = A \rightarrow \Lambda' \rightarrow 0$$

\textit{satisfying}

1. $C_1 = \sum A u_j$ with $d_1 u_j = r_{(p_j, 0, 0, \cdots)}$;
2. $C_i = \prod A w^i_0$ is locally finitely generated as an $A$-module, $i > 1$;
3. $\ker(Jd_i) \subset j(\operatorname{im} d_{i+1})$ in $JC_i$ for all $i$, $n \geq 0$.

Any such $C$ is an $A$-projective resolution of $\Lambda'$.

The proof is straightforward. Notice that the infinite direct product $\prod A w^0_i$ is not necessarily free over $A$; it is projective, however. As a further aid to computation there is

**Lemma 4.** If \{\(C_i, d_i\)\} is any $A$-projective resolution of $\Lambda'$, write $C^*_i = \operatorname{Hom}_A(C_i, \Lambda')$, $d^*_i = \operatorname{Hom}_A(d_i, \Lambda')$. Then

$$\operatorname{Ext}^s_A(\Lambda', \Lambda') = \operatorname{Tors}(\operatorname{cok}(d^*_t)), \quad (s, t) \neq (0, 0).$$

**Proof.** This follows from the fact that $\operatorname{Ext}^s_A$ is finite for $(s, t) \neq (0, 0)$ [7, Corollary 2.1].

Thus in determining Ext we need know just the boundaries, and not the cycles too. In fact we can even work over $Z_{p^f}$ for suitable $f$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Now we can prove

**Proposition 5.** \( \text{Ext}^{0,t} = 0 \) unless \( t = 0 \); \( \text{Ext}^{0,0} = \mathbb{Z} \).

**Theorem 6.** \( \text{Ext}^{2,t} \) contains a direct summand isomorphic to \( \mathbb{Z}_p \) for \( t = 2p^i(p-1) \) (\( i \geq 1 \)) and \( t = 2(p^i+1)(p-1) \) (\( i > 1 \)).

**Theorem 7.** For \( p = 2 \), the element of \( \text{Ext}^{2,t} \) found in Theorem 6 maps to the Arf-invariant element \( h_2 \) of the classical Adams spectral sequence [5].

**Proof.** Apply the Thom map (Proposition 1) to a suitable \( A \)-resolution.

**Proposition 8.** The two-primary part \( \text{Ext}^{*,*} \) has the following "edge" values:

\[
\begin{align*}
\text{Ext}^{n,2(n+k)} & = 0, & k < 0; \\
& = \mathbb{Z}_2, & k = 0, n \geq 1 \text{ (generated by } h^n) ; \\
& = 0, & k = 1, n \geq 2; \\
& = \mathbb{Z}_2, & 2 \leq k \leq 5, n \geq 4 \text{ (generated by } h^{n-1}a_1a_1+1).
\end{align*}
\]

Further computations of the additive structure of \( \text{Ext}^{*,*} \) in low dimensions are given in Figure 1. Thanks to Proposition 8, the first three nonzero Novikov differentials \( d_3a_i = h^3a_{i-1} \), \( i = 3, 6, 7 \), give rise to infinite towers of nonzero \( d_i \)'s. Moreover, every other differential in the range \( t - s \leq 17 \) must be zero for dimensional reasons. Finally, \( \text{Ext}^{*,*} \) has a vanishing line considerably lower than that of the \( E_\infty \)-term of the classical Adams spectral sequence in this range of dimensions. We conjecture that the preceding four sentences are also true without restriction on the dimensions.

Similar computations for \( p = 3 \) disclose striking edge properties like Proposition 8, but many fewer differentials. Contrary to Novikov's conjecture [7], there is a nonzero differential \( d_3: E_2^{3,30} \rightarrow E_7^{7,40} \) for \( p = 3 \). This differential, whose existence is inferred from Toda's result [10], also gives rise to an infinite family of nonzero differentials. It is encouraging that there is only one nonzero differential in the range \( t - s \leq 40 \), as compared to 17 in the classical 3-primary Adams spectral sequence.

**References**

2. ———, *On the groups \( J(X) \)*. II, Topology **3** (1965), 137–173; IV, ibid. **5** (1966), 21–71. MR **33** #6626; 6628.

4. ———, *Quillen's work on formal group laws and complex cobordism*, University of Chicago Lecture Notes Series, 1970.


**Figure 1.** $\text{Ext}^{k,1}$ for the Novikov sequence.