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ON THE HOMOLOGY OF A FIXED POINT SET

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The objects to be studied are the continuous functions of the sort $g: X\times T\to X$. Here X is an ANR(M) (for metric spaces) such as a manifold or function space, and T is any normal Hausdorff space, but which in nature would be acyclic [11], or a semigroup [4], [6], and [14]. For an open subset O of $X\times T$ define a *fixed point* of $g|O$ to be a point x such that there exists (x, t) in O which is a solution to $g(x, t) =x$. The closure of the set of fixed points of $g|O$ is Fix $g|O$, and the set of solutions (x, t) in O is $S(g|O)$. It will generally be assumed that the closure of $g(O)$ is compact and that $S(g|O)$ is closed in $X\times T$. These conditions will be signaled by the terminology "g is nondegenerate on O." Then there is a homomorphism induced by g,

$$\theta_*(g): H_*(T, T_0) \to \check{H}_*(\mathrm{Fix}\ g \mid O),$$

where $X\times T_0$ is disjoint from $S(g|O)$. This homomorphism generalizes the Leray θ-homomorphism of rings of pseudocycles [11, Chapter VII] and the index cycle of Fuller [4, p. 135]; however these rela-

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tionships are to be discussed elsewhere [9]. Here there will be given a global formula for $\theta_*(g)$ analogous to the Lefschetz trace formula, and it will be applied to obtain, in a special case, information about the Čech homology of Fix g which is a homotopy invariant of g.

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1. **The generalized Lefschetz formula.** Let g be nondegenerate on $O=X\times T$ and let K be a compact subset of X which contains the image of g. Since X is an ANR(M) the inclusion $i:K\subset X$ induces a homomorphism of Čech homology groups, i_*, which has a finite dimensional image (coefficients are always taken in a fixed field Q). Choose for each $n\geqq 0$ a finite basis $\{i_*(z_n^j)\}_j$ for the image of $(i_*)_n$, and let $\{z^n_j\}_j$ be a family of linearly independent elements of the Čech cohomology $\check{H}^n(X)$ orthonormal to $\{i_*(z_n^j)\}_j$. Then for any $z_m\in H_m(T)$ and each $i_*(z_n^j)$ there is a unique expression

$$g_*(i_*(z_n^j)\otimes \check{z}_m)=\sum_k b_{jk}z^k_{n+m},\qquad b_{jk}\in Q.$$

The coefficients b_{jk} depend on $\check{z}_m$ as well as j and k. Here $\check{z}_m$ is the natural image of z_m in $\check{H}_m(T)$ under the natural transformation $H_*\to\check{H}_*$.

Definition. *The Λ-homomorphism induced by g of $H_*(T)$ into $\check{H}_*(K)$ is defined at $z_m\in H_m(T)$ as*

$$\Lambda_*(g)(z_m)=\sum_{jk}(-1)^n b_{jk}i^*(z^n_j)\cap z^k_{n+m}$$

where $\cap$ denotes cap product.

Comment. To see the relationship of $\Lambda_*(g)$ to the Lefschetz trace formula, take $m=0$. If T is a point and X is connected so that $Q=H_0(T)=H_0(X)$, the Lefschetz number of g is $\Lambda_*(g)(1)$ where 1 is the unity of Q [7] and [10].

Theorem 1. *Let g be nondegenerate on $X\times T$ with an image which is contained in a compact set K. Let j:Fix $g\subset K$ be inclusion. Then $\Lambda_*(g)=j_*\circ\theta_*(g)$.*

2. **Localization of** $\Lambda_*(g):\theta_*(g|O)$. Except for the use of Čech homology $\check{H}_*$ as well as singular homology H_*, this section is patterned on the elegant treatment [2] of the local fixed point index of Leray. For the moment then, X is an open subset of a Euclidean space, $\boldsymbol{R}^n$, of dimension n, $g:X\times T\to\boldsymbol{R}^n$ is nondegenerate on an open subset O of $X\times T$, and the image of $g|O$ is contained in a compact set

K. An orientation is chosen for $\boldsymbol{R}^n$ and this determines [2] a fundamental class $\mathcal{O}_K \in H_n(\boldsymbol{R}^n, \boldsymbol{R}^n - K)$. Let

$$(\pi - g) \times g_K : (O, O - S(g|O)) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\}) \times K$$

be defined by $(\pi - g) \times g_K(x, t) = (x - g(x, t), g(x, t)), (x, t) \in O$.

Let $T_0 \subset T$ be such that $X \times T_0$ is disjoint from $S = S(g|O)$. Denote Fix $g|O$ by F. For $Z_m \in H_m(T, T_0)$, (singular homology) $\mathcal{O}_F \times Z_m$ is an element of $H_*[(\boldsymbol{R}^n, \boldsymbol{R}^n - F) \times (T, T_0)]$, and $i_*(\mathcal{O}_F \times Z_m) \in H_*(\boldsymbol{R}^n \times T, \boldsymbol{R}^n \times T - S)$, where i_* is induced by inclusion. By excision we may regard $i_*(\mathcal{O}_F \times Z_m)$ as an element of $H_*(O, O - S)$, so that $[(\pi - g) \times g_K]_* i_*(\mathcal{O}_F \times Z_m)$ is a well-defined element of $H_*[(\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\}) \times K]$. Thus we have defined a homomorphism

$$[(\pi - g) \times g_K]_* \circ i_* \circ (\mathcal{O}_F \times \cdot) : H_*(T, T_0) \to H_*[(\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\}) \times K].$$

Evidently this does not depend on the choice of open set O which contains S as long as K contains $g(O)$. Also if $j : K \subset K'$, then

$$j_*[(\pi - g) \times g_K]_* \circ i_* \circ (\mathcal{O}_F \times \cdot) = [(\pi - g) \times g_{K'}]_* \circ i_* \circ (\mathcal{O}_F \times \cdot).$$

As a consequence there exists a homomorphism

$$[(\pi - g) \times g_F]_* \circ i_* \circ (\mathcal{O}_F \times \cdot) : H_*(T, T_0) \to \check{H}_*[(\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\}) \times F]$$

which is defined to be the inverse limit of the system of homomorphisms

$$\{[(\pi - g) \times g_K]_* \circ i_* \circ (\mathcal{O}_F \times \cdot) : K \text{ is a compact neighborhood of } F\}.$$

DEFINITION. Let $\theta_*(g|O) = (\mathcal{O}_{\{0\}} \times \cdot)^{-1}[(\pi - g) \times g_F]_* \circ i_* \circ (\mathcal{O}_F \times \cdot)$. Then

$$\theta_*(g|O) : H_*(T, T_0) \to \check{H}_*(F).$$

PROPOSITION 2.1. *$\theta_*(g|O)$ depends only on the term of g at $S(g|O)$.*

PROOF. This is an immediate consequence of its definition.

The techniques of [2] provide elementary proofs for the next four properties of $\theta_*(g|O)$: Additivity, Multiplicativity, Naturality and Homotopy Invariance.

Additivity. If O is the disjoint union of open sets O_1 and O_2, then

$$\check{H}_*(\text{Fix } g|O) = \check{H}_*(\text{Fix } g|O_1) \oplus \check{H}_*(\text{Fix } g|O_2)$$

and

$$\theta_*(g|O) = \theta_*(g|O_1) \oplus (g|O_2).$$

Multiplicativity. Let $g : X \times T \to \boldsymbol{R}^n$ and $g' : X' \times T' \to \boldsymbol{R}^{n'}$ be as in the definition of θ_* with g nondegenerate on $O \subset X \times T$ and g' nondegenerate on $O' \subset X' \times T'$. Then

$$\text{Fix } g \times g' | O \times O' = (\text{Fix } g | O) \times (\text{Fix } g' | O')$$

and

$$\theta_*(g \times g' | O \times O') = \theta_*(g | O) \otimes \theta_*(g' | O').$$

Naturality (*in* T). Suppose that g is nondegenerate on $O \subset X \times T$ and $f: T' \to T$ is a map. Let $O' = (1 \times f)^{-1}(O)$. Then $g \circ (1 \times f)$ is nondegenerate on O'. Let $f(T_0') \subset T_0$. Then

$$\text{Fix } g \circ (1 \times f) | O' = \text{Fix } g | O$$

and

$$\theta_*(g \circ (1 \times f) | O') = \theta_*(g | O) \circ f_*: H_*(T', T_0') \to \check{H}_*(\text{Fix } g | O).$$

Homotopy Invariance. Suppose that $g_s: X \times T \to \boldsymbol{R}^n$, $0 \leqq s \leqq 1$, is a homotopy. It is said to be *nondegenerate* on $O \subset X \times T$ if the map $(x, t, s) \to g_s(x, t)$ is nondegenerate on $O \times I$, and its *fixed point set* is $F = \text{Cl}(\bigcup_s \text{Fix } g_s | O)$. Let $i_s: \text{Fix } g_s | O \subset F$ be inclusion. Then

$$(i_0)_* \circ \theta_*(g_0 | O) = (i_1)_* \circ \theta_*(g_1 | O).$$

Next to the existence of a global index, the commutativity property provides one of the most important aids to the computation of a fixed point index, and one which also enables one to extend θ_* to ANR(M) spaces X. In Theorem 2 is given a form of this property sufficient to accomplish this extension, using the techniques of [2] for Euclidean neighborhood retraits and of [3], [12, Theorem 2] and [16] for ANR(M) spaces.

THEOREM 2. (COMMUTATIVITY). *Suppose given* $g: X \times T \to \boldsymbol{R}^n$, *as in the definition of* $\theta_*(g | O)$, *and given* $X' \subset \boldsymbol{R}^{n'}$ *open, and a map* $r: X' \to X$ *and compact sets* $K \subset \boldsymbol{R}^n$, $K' \subset \boldsymbol{R}^{n'}$ *such that* $K \supset g(O)$, *and* $r | K'$ *is a homeomorphism of* K' *onto* K. *Let* $j = (r | K')^{-1}$, *and let* $O' = (r \times 1)^{-1} O$. *Then* $j \circ g \circ r \times 1$ *is nondegenerate on* O' *and* $\theta_*(j \circ g \circ r \times 1 | O') = j_* \theta_*(g | O)$.

SKETCH OF PROOF. The simplest case (case 1) occurs when r is an orthogonal projection of $\boldsymbol{R}^{n'}$ onto $\boldsymbol{R}^n$. This case is computational and left to the reader. Case 2 is when $n' \leqq n$ and r is inclusion of $\boldsymbol{R}^{n'}$ in $\boldsymbol{R}^n$. This is reducible to case 1 by a homotopy $g \circ (h_s \times 1)$, $0 \leqq s \leqq 1$, where $h_s: \boldsymbol{R}^n \to \boldsymbol{R}^n$, $0 \leqq s \leqq 1$, is the linear homotopy of the identity to the orthogonal projection of $\boldsymbol{R}^n$ to $\boldsymbol{R}^{n'}$. The homotopy property thus obtains case 2. In the general case one may assume that X and X' are closed polyhedral neighborhoods in $\boldsymbol{R}^n$ and $\boldsymbol{R}^{n'}$ of K and K', respectively, which do not contain the origin. The mapping cylinder $\boldsymbol{Z}_r$ of r is the set in $\boldsymbol{R}^n \oplus \boldsymbol{R}^{n'}$ of points either of the form $x \oplus 0$ or of the form $r(x') \oplus (1-a)x'$, $x' \in X'$, $0 \leqq a \leqq 1$. Likewise the mapping cylinder $\boldsymbol{Z}_j$ is defined and is contained in $\boldsymbol{Z}_r$. Now g and $j \circ g \circ r \times 1$ have the obvious extensions to $\boldsymbol{Z}_r$ (and to a neighborhood of $\boldsymbol{Z}_r$ since the latter is an ANR) and these extensions are homotopic by the linear homotopy that moves K to K' through $\boldsymbol{Z}_j$. By homotopy and case 2, Theorem 2 is proven.

3. **Agreement with $\Lambda_*(g)$.** As in paragraph 1, $g: X \times T \to X$ is a map with image contained in a compact set K, j: Fix $g \subset K$ is inclusion and X is an ANR. As usual [2], it may be assumed that X is in fact an open subset V of $\boldsymbol{R}^n$. Let $d: \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be defined by $d(x, y) = x - y$. Then d maps $(V, V-K) \times K$ into $(\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\})$ and induces

$$d_*: H_*[(V, V-K) \times K] \to H_*(\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\}).$$

By the Künneth rule $H_*[(V, V-K) \times K] = H_*(V, V-K) \otimes H_*(K)$. Define $\hat{d}_*: H_*(V, V-K) \to [H_*(K)]^*$ as in [2]; $\hat{d}_*(v)(k) = d_*(v \otimes k)$. For a graded Q-module (that is vector space) M, define

$$\Theta_M: H_*(V, V-K) \otimes M \to \operatorname{Hom}(H_*(K), M)$$

by the rule $\Theta_M(v \otimes m)k = (-1)^{|m||k|}(\hat{d}_*(v)(k))m$, where "$|\cdot|$" means "dimension of." A thorough discussion of this sign convention is given in [13, I §1] and its understanding is necessary for what follows. By rephrasing Lemma 4.2 of [2] one obtains

LEMMA 3.1. *If $i: K \subset V$ is inclusion, then Θ_{HV} transforms $\Delta_*(\mathcal{O}_K)$ into i_*. Here the subscript HV means $H_*(V)$.*

LEMMA 3.2. *If K is a finite polyhedron, then for any M, Θ_M is a natural equivalence. In particular $\hat{d}_* = \Theta_Q$ is a natural equivalence.*

PROOF. By excision we may assume V is so small that there is a retraction $r: V \to K$. Then $r_* \circ \Theta_{HV}(\Delta_*(\mathcal{O}_K)) = r_* \circ i_* =$ identity of $H_*(K)$. But Θ_M is natural in M, and it follows that $\Theta_{HK}[(1 \otimes r_*) \circ \Delta_*(\mathcal{O}_K)]$ is the identity of $H_*(K)$. Invoking naturality again, it follows that Θ_{HK} is an epimorphism. But $H_*(V, V-K)$ and $H^*(K)$ have the same dimension (Poincaré duality). Thus $H_*(V, V-K) \otimes H_*(K)$ and Hom $(H_*(K), H_*(K))$ have the same finite dimension and Θ_{HK} must be an isomorphism. But Θ is a natural transformation implying then that Θ is a natural equivalence, as claimed.

Let K be a polyhedron in $\boldsymbol{R}^1$, and let $d \times \pi_K: (V, V-K) \to (\boldsymbol{R}^n, \boldsymbol{R}^n - \{0\}) \times K$ be defined by $d \times \pi_K(x, y) = (x-y, y)$. Let Δ:

$K \to K \times K$ be the diagonal map. Then the cap product of $\mathcal{Z}^p \in [H_*(K)]^*$ with $\mathcal{Z}_{p+m} \in H_*(K)$ is $\mathcal{Z}^p \cap \mathcal{Z}_{p+m} = (e \otimes 1)(\mathcal{Z}^p \otimes \Delta_*(\mathcal{Z}_{p+m}))$ where $e(\mathcal{Z}^p \otimes \mathcal{Z}_p) = \mathcal{Z}^p(\mathcal{Z}_p)$ is the evaluation. If we use $\hat{d}_*$ to identify $H_*(V, V-K)$ with $[H_*(K)]^*$, then e becomes d_* and one has that $\cap$ becomes $(d \times \pi_K)_*$. Formally, there is the

LEMMA 3.3. *If K is a finite polyhedron, and $\mathcal{Z}^n \in H^*(K)$, $\mathcal{Z}_{n+m} \in H_*(K)$, then*

$$\mathcal{Z}^n \cap \mathcal{Z}_{n+m} = (\mathcal{O}_K \times \cdot)^{-1}(d \times \pi_K)_*(\hat{d}_*^{-1}(\mathcal{Z}^n) \times \mathcal{Z}_{n+m}).$$

PROOF OF THEOREM 1. We may assume $X = V$ is open in $\boldsymbol{R}^n$. Since Fix g is the intersection of polyhedral neighborhoods, its Čech homology is the inverse limit of such neighborhoods. We may thus assume that K is a polyhedron and $H_*(K) = \check{H}_*(K)$. Let $i: K \to V$ be inclusion. For $a \otimes b \otimes \mathcal{Z}_m$ in $H_*(V, V-K) \otimes H_*(V) \otimes H_m(T)$, the naturality of Θ implies that

$$\Theta_{HK}[(1 \otimes g_*)(a \otimes b \otimes \mathcal{Z}_m)] = \hat{g}_*(\mathcal{Z}_m) \circ \Theta_{HV}(a \otimes b),$$

where $\hat{g}_*(\mathcal{Z}_m): H_*(V) \to H_*(K)$ is defined as $\hat{g}_*(\mathcal{Z}_m)(b) = (-1)^{m|b|} \cdot g_*(b \otimes \mathcal{Z}_m)$. But then

$$(1 \otimes g_*)(\Delta_*(\mathcal{O}_K) \otimes \mathcal{Z}_m) = \overset{-1}{\Theta}_{HK}(\hat{g}_*(\mathcal{Z}_m) \circ i_*),$$

by Lemma 3.1. Applying cap product to the left side of this equality yields by 3.3, $\theta_*(g)(\mathcal{Z}_m)$, and on the right side it yields by direct calculation $\Lambda_*(g)(\mathcal{Z}_m)$. This proves Theorem 1.

THEOREM 3 (CHANGED IN PROOF). *Suppose that X is a connected compact polyhedron and $H_k(X) = 0$ for $k > n$. Let $g: X \times X \to X$ be a multiplication with left homotopy identity. Then $H_n(X)$ is a direct summand of H_n(Fix g).*

PROOF. From the existence of left homotopy identity one computes that $\Lambda_*(g)$ is the identity on $H_n(X)$. From Theorem 1, $H_n(X)$ is a direct summand of H_n(Fix g).

ADDED IN PROOF. If $X = M^n$ is a closed n-manifold, Theorem 3 implies that Fix $g = M^n$. That was the original form of my Theorem 3, however, R. F. Brown pointed out to me that Fix $g = M^n$ may be easily obtained by degree theory.

REFERENCES

1. F. E. Browder, *On the continuity of fixed points under deformations of continuous mappings*, Summa Brasil. Mat. **4** (1960), 183–190. MR **24** #A543.

2. A. Dold, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology **4** (1965), 1–8. MR **33** #1850.

3. J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), 353–367. MR **13**,373.

4. F. Fuller, *An index of fixed point type for periodic orbits*, Amer. J. Math. **89** (1967), 133–148. MR **35** #497.

5. A. Granas, *Topics in the fixed point theory*, Séminaire Jean Leray Paris, 1969/70.

6. J. K. Hale, *Geometric theory of functional-differential equations*, Proc. Internat. Sympos. Differential Equations and Dynamical Systems (Mayaguez, P.R., 1965), Academic Press, New York, 1967, pp. 247–266. MR **36** #5465.

7. H. Hopf, *Über die algebraische Anzahl von Fixpunkten*, Math. Z. **29** (1929), 493–524.

8. R. Knill, *A local fixed point index for compact maps with noncompact domains*, Séminaire Jean Leray Paris, 1969/70.

9. ——— *A general setting for fixed point theory* (in prep.)

10. S. Lefschetz, *Intersections and transformations of complexes and manifolds*, Trans. Amer. Math. Soc. **28** (1926), 1–49.

11. J. Leray, *Sur les équations et les transformations*, J. Math. Pures. Appl. (9) **24** (1945) 201–248.

12. M. Nagumo, *Degree of mapping in convex linear topological spaces*, Amer. J. Math. **73** (1951), 497–511. MR **13**,150.

13. Weishu Shih, *Homologie des espaces fibrés*, Inst. Hautes Études Sci. Publ. Math. No. 13 (1962). MR **26** #1893.

14. S. Smale, *Differentiable dynamical systems*. Bull. Amer. Math. Soc. **73** (1967), 747–817. MR **37** #3598.

15. E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966. MR **35** #1007.

16. M. Wojdyslawski, *Rétractés absolus et hyperespaces des continus*, Fund. Math. **32** (1939), 184–192.

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