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ON THE DIVERGENCE OF MULTIPLE FOURIER SERIES

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Carleson showed in [1] that the partial sums of the Fourier series of a function $f\in L^2[0, 2\pi]$ converge almost everywhere. In this note and [2] we take up some of the n-dimensional generalizations of the almost everywhere convergence theorem. As usual in questions of multiple Fourier series, the answers depend on exactly what we mean by a "partial sum". For instance, suppose $f\in L^2([0, 2\pi]\times[0, 2\pi])$ has the double Fourier series $f(x, y)\sim\sum_{m,n=-\infty}^{\infty} a_{mn}e^{i(mx+ny)}$. Which of the following hold almost everywhere?

$$\text{(A)} \qquad f(x, y) = \lim_{M,N\to\infty} \sum_{|m|\leq M;\ |n|\leq N} a_{mn}e^{i(mx+ny)}.$$

$$\text{(B)} \qquad f(x, y) = \lim_{M\to\infty} \sum_{|m|,|n|\leq M} a_{mn}e^{i(mx+ny)}.$$

$$\text{(C)} \qquad f(x, y) = \lim_{R\to\infty} \sum_{m^2+n^2\leq R} a_{mn}e^{i(mx+ny)}.$$

We shall exhibit below a counterexample to (A). In [2], [3], [4] it is shown that (B) and some of its variants are simple consequences of Carleson's one-dimensional result. Problem (C) is still open.

THEOREM 1. *There is a continuous function on* $[0, 2\pi]\times[0, 2\pi]$ *for which* (A) *holds nowhere.*

To simplify our exposition, we shall first prove a slightly weaker result, and then explain how to modify the proof to obtain Theorem 1.

THEOREM 1′. *There is a continuous function on* $[0, 2\pi]\times[0, 2\pi]$ *for which* (A) *does not hold anywhere in the square* $Q=\{(x, y)\mid 1/10\leqq x, y\leqq 2\pi-1/10\}$.

PROOF OF THEOREM 1′. The main idea in our proof is to study the partial sums of the double Fourier series of the function $f_\lambda(x, y) = e^{i\lambda xy}$ defined on $[0, 2\pi]\times[0, 2\pi]$. To do so, we use the Dirichlet formula: For $f(x, y)\sim\sum_{m,n=-\infty}^{\infty} a_{mn}e^{i(mx+ny)}$ defined on $[0, 2\pi]\times[0, 2\pi]$, we have

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$$(1)\qquad \begin{aligned} S_{MN}f(x, y) &\equiv \sum_{|m|\leq M;\ |n|\leq N} a_{mn}e^{i(mx+ny)} \\ &= \frac{1}{\pi^2}(T_{MN}f(x, y) - T_{-MN}f(x, y) \\ &\qquad - T_{M-N}f(x, y) + T_{-M-N}f(x, y)) + \text{Error terms} \end{aligned}$$

where, by definition

$$T_{MN}f(x, y) = e^{i(Mx+Ny)}(\text{P.V.}) \int_0^{2\pi}\int_0^{2\pi} \frac{e^{-i(Mx'+Ny')}}{(x-x')(y-y')} f(x', y')\, dx'\, dy'$$

for any real numbers M and N. (See [5, Chapter 17, formula 1.12].)

The reader may easily check that for $f=f_\lambda$ and for $(x, y)\in Q$, the error terms in (1) are $O(1)$, and consequently of no importance in what follows.

LEMMA 1. *Given any $(x, y)\in Q$, we can find integers M and N, for which $|T_{MN}f_\lambda(x, y)|$ is as large as* $\log \lambda$.

To prove the lemma, we take $M=[\lambda y]$ and $N=[\lambda x]$, brackets denoting the greatest integer function. By making straightforward estimates, we can easily check that changing $M=[\lambda y]$, $N=[\lambda x]$ to $M=\lambda y$, $N=\lambda x$ alters $T_{MN}f_\lambda(x, y)$ by at most $O(1)$. So to prove our lemma, we have only to show that

$$(2)\qquad \begin{aligned} |T_{\lambda y,\lambda x}f_\lambda(x, y)| &= \left|\int_0^{2\pi}\int_0^{2\pi} \frac{e^{-i(\lambda yx'+\lambda xy')}}{(x-x')(y-y')} e^{i\lambda x'y'}\, dx'\, dy'\right| \\ &= \left|\int_0^{2\pi}\left[\int_0^{2\pi} \frac{e^{i\lambda(x-x')(y-y')}}{(x-x')(y-y')}\, dx'\right] dy'\right| \end{aligned}$$

is large. The inner integral on the right-hand side has the form

$$(3)\qquad \int_{-A}^{B} \frac{e^{i\xi t}}{t}\, dt$$

and consequently satisfies properties

$$(4)\qquad \int_0^{2\pi} \frac{e^{i\lambda(x-x')(y-y')}}{(x-x')}\, dx' = \frac{\pi i}{2}\operatorname{sgn}(y-y') + O\left(\frac{1}{\lambda|y-y'|}\right) \quad \text{for } (x, y)\in Q$$

and

$$(5)\quad \int_0^{2\pi} \frac{e^{i\lambda(x-x')(y-y')}}{(x-x')}\,dx' = C_{xy} + O(\lambda\,|\,y-y'\,|)$$
$$\text{for } (x, y) \in Q,\ |\,C_{xy}\,| = O(1).$$

These are just well-known facts about the Dirichlet integral (3), heavily camouflaged by notation.

Using (4) and (5) we can evaluate the right-hand side of (2). In fact,

$$\int_0^{2\pi} \frac{1}{(y-y')}\left[\int_0^{2\pi} \frac{e^{i\lambda(x-x')(y-y')}}{(x-x')}\,dx'\right]dy'$$
$$= \int_0^{y-1/\lambda} + \int_{y-1/\lambda}^{y+1/\lambda} + \int_{y+1/\lambda}^{2\pi} \equiv T^1 + T^2 + T^3.$$

Using (4), we get

$$(6)\qquad T^1 = \frac{\pi i}{2}\log\lambda + O(1) \quad\text{and}\quad T^3 = \frac{\pi i}{2}\log\lambda + O(1).$$

(5) shows that $T^2 = O(1)$. Therefore, by equation (2)

$$|\,T_{MN}f_\lambda(x, y)\,| = |\,T^1 + T^2 + T^3\,| + O(1) = \pi\log\lambda + O(1).$$

This proves the lemma.

The point is that we were able to cook up the signs so that the expected cancellation between T^1 and T^3 does not occur.

LEMMA 2. *For* $(x, y) \in Q$, $M = [\lambda y]$, $N = [\lambda x]$, *the quantities* $T_{-MN}f_\lambda(x, y)$, $T_{M-N}f_\lambda(x, y)$, *and* $T_{-M-N}f_\lambda(x, y)$ *are all* $O(1)$.

The proof is just a computation copying the one we just did for Lemma 1. This time, however, the signs are not arranged as well as before, and the terms analogous to T^1 and T^3 nearly cancel. Details are left to the reader.

Putting Lemmas 1 and 2 into formula (1) yields our basic inequality: For f_λ, (x, y), and M, N as above, $|S_{MN}f_\lambda(x, y)| \geqq c\log\lambda$. That is, the partial sums of the double Fourier series of f_λ grow as large as $\log\lambda$ throughout Q, despite the fact that $\|f_\lambda\|_\infty = 1$.

Now it is a routine task to form an infinite series $F = \sum_{k=1}^{\infty} c_k f_{\lambda_k}$, in the spirit of the proof of the uniform boundedness theorem, so that $\limsup_{M,N\to\infty} |\,S_{MN}F(x, y)\,| = \infty$ for every $(x, y) \in Q$, and yet F is continuous on $[0, 2\pi] \times [0, 2\pi]$. For example, we can use $c_k = 1/2^{2k}$ and $\lambda_k = 2^{4^{2^k}}$. Q.E.D.

Next we sketch how to improve our construction to obtain Theorem 1. Let $\varphi(t)$ be a C^∞ function on R^1, equal to zero near $t=0$ or 2π, and equal to one for $1/20 \leqq t \leqq 2\pi - 1/20$. Then replace f_λ by $h_\lambda(x, y) \equiv \varphi(x)\varphi(y)e^{i\lambda xy}$. Lemmas 1 and 2 go through unscathed for h_λ. However, h_λ is not merely continuous on $[0, 2\pi] \times [0, 2\pi]$; it is also continuous on the torus T^2 obtained from $[0, 2\pi] \times [0, 2\pi]$ by identifications. Now instead of $\sum c_k f_{\lambda_k}$ we form the infinite series $F(x, y) = \sum_{k=1}^{\infty} c_k h_{\lambda_k}((x, y) \cdot T_k)$, where $\{c_k\}$ and $\{\lambda_k\}$ are as before, and $\{T_k\}$ is a sequence of translations on the torus. If the c's, λ's, and T's are properly picked, F will be a continuous function on the 2-torus, for which (A) holds nowhere.

Our counterexample actually proves more than Theorem 1. Specifically, recall the classical theorem of Kolmogoroff-Seliverstoff-Plessner, which states that the nth partial sum of the Fourier series of an L^2 function on $[0, 2\pi]$ is $o((\log n)^{1/2})$ almost everywhere, as $n \to \infty$. (See [5, Chapter 13, Theorem 1.2].) The proof in [5] generalizes easily to k dimensions and yields, in the two-dimensional case, $\max_{M,N \leq R} |S_{MN}f(x, y)| = o(\log R)$ almost everywhere. Carleson's theorem in one dimension provides a dramatic improvement, from $o((\log n)^{1/2})$ to $O(1)$. However, our counterexample shows that already in two dimensions the Kolmogoroff-Seliverstoff-Plessner theorem essentially cannot be improved. (Nevertheless, see [3] for an interesting refinement.) The same is true in any even number of dimensions, as follows from repeatedly crossing our counterexample with itself. In an odd number of dimensions, the question of the growth of partial sums is not completely settled, since the classical result is

$$\max_{M_1, \ldots, M_k \leqq R} |S_{M_1 \cdots M_k} f| = o((\log R)^{k/2}) \quad \text{almost everywhere,}$$

while our example shows only that we cannot hope for more than

$$\max_{M_1, \ldots, M_k \leqq R} |S_{M_1 \cdots M_k} f| = o((\log R)^{(k-1)/2}).$$

Note that our counterexample also disproves

$$\text{(A*)} \qquad \lim_{M,N \to \infty; 1/2 < M/N < 2} S_{MN}f(x, y) = f(x, y).$$

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