BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 2, March 1971

THE STRUCTURE OF ω-REGULAR SEMIGROUPS

BY JANET AULT AND MARIO PETRICH

Communicated by R. S. Pierce, July 20, 1970

1. Finding the complete structure of regular semigroups of a certain class has succeeded only when sufficiently strong conditions on idempotents and/or ideals have been imposed. On the one hand, there is the theorem of Rees [7], giving the structure of completely 0-simple semigroups, and its successive generalizations to primitive regular semigroups [2], and $\mathfrak{J}$- and $\mathfrak{J}_1$-regular semigroups [4]. On the other hand, with very different restrictions, Reilly [8] has determined the structure of bisimple ω-semigroups, Kočin [1] of inverse simple ω-semigroups, Munn [5] of inverse ω-semigroups.

An ω-chain with zero is a poset $\{e_i \mid i \geqq 0\} \cup 0$ with $e_i > e_j$ if $i < j$, and $0 < e_i$ for all i, j. We call a regular semigroup S *ω-regular* if S has a zero and the poset of its idempotents is an orthogonal sum [2] of ω-chains with zero. We announce here the complete determination of the structure of such semigroups, including various special cases thereof, and briefly mention their isomorphisms.

2. An ω-regular semigroup can be uniquely written as an orthogonal sum of ω-regular prime (i.e., with 0 a prime ideal) semigroups. This reduces the problems of structure and isomorphism to ω-regular prime semigroups. We distinguish three cases: (i) 0-simple, (ii) prime with a proper 0-minimal ideal, (iii) prime without a 0-minimal ideal. Case (i) is the most difficult (and interesting) and includes a variety of special cases some of which reduce to those constructed by Reilly [8], Kočin [1], and Munn [5], [6].

3. Let A be a nonempty set, d be a positive integer, V be a semigroup which is a chain of d groups $G_0 > G_1 > \cdots > G_{d-1}$, and σ be a homomorphism of V into G_0. Let $w: A \to \{0, 1, \cdots, d-1\}$ be any function, denoted by $w: \alpha \to w_\alpha$. For $\alpha \in A$, $0 \leqq i, j < d$, define $\langle \alpha, i \rangle$ by

$$\langle \alpha, i \rangle \equiv w_\alpha + i \pmod{d}, \qquad 0 \leqq \langle \alpha, i \rangle < d,$$

and define $[i, \alpha, j]$ to satisfy

$$[i, \alpha, j]d = (i - j) - (\langle \alpha, i \rangle - \langle \alpha, j \rangle).$$

AMS 1970 *subject classifications*. Primary 20M10.

Key words and phrases. Regular semigroups, ω-regular, ω-semigroups, 0-simple, 0-bisimple, prime semigroups, orthogonal sum, Brandt semigroups, primitive inverse semigroups, 0-restricted homomorphisms, semilattice, tree, Brandt tree, rooted Brandt tree.

Construction I. On the set

$$S = \{(\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, m, n \geqq 0, g \in V\} \cup 0,$$

define a multiplication by: for $g_i \in G_i$, $g_j \in G_j$, $v = n - s - [i, \beta, j]$,

$$(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m - [i, \alpha, j] - v, (g_i\sigma^{-v})g_j, t, \gamma) \quad \text{if } v < 0, \text{ or } v = 0, i \leqq j;$$

$$(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m, g_i(g_j\sigma^v), t + [i, \gamma, j] + v, \gamma) \quad \text{if } v > 0, \text{ or } v = 0, i > j;$$

and all other products are equal to 0. The set S with this multiplication will be denoted by $\mathfrak{O}(A, w; V, \sigma)$.

Construction II. On the set

$$S' = \{(\alpha, m, g, n, \beta) \mid \alpha, \beta \in A, m - w_\alpha \equiv n - w_\beta \equiv i \pmod d, g \in G_i\} \cup 0,$$

define a multiplication by: for $g_i \in G_i$, $g_j \in G_j$, $v = n' - s' - [i, \beta, j]$, where $n = n'd + n''$, $s = s'd + s''$, $0 \leqq n''$, $s'' < d$,

$$(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m + s - n, (g_i\sigma^{-v})g_j, t, \gamma) \text{ if } n \leqq s;$$

$$(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m, g_i(g_j\sigma^v), t + n - s, \gamma) \quad \text{if } n > s;$$

and all other products are equal to 0. The set S' with this multiplication will be denoted by $\mathfrak{O}[A, w; V, \sigma]$.

The following is our fundamental result.

THEOREM 1. *For a groupoid S, the following statements are equivalent.*
(i) *S is a 0-simple ω-regular semigroup;*
(ii) *S is isomorphic to $\mathfrak{O}(A, w; V, \sigma)$;*
(iii) *S is isomorphic to $\mathfrak{O}[A, w; V, \sigma]$.*

The proof of "(i)⇒(ii)" consists of "introducing coordinates" into various $\mathfrak{L}$- and $\mathfrak{R}$-classes and of constructing the homomorphism σ; it is quite long and is broken into a sequence of lemmas. For "(ii)⇒(iii)" one finds a suitable isomorphism, while "(iii)⇒(i)" consists of a verification of the defining properties of a 0-simple ω-regular semigroup.

Define the *top* of S in the theorem by

$$\mathfrak{I}(S) = \{a \in S \mid e \,\mathfrak{L}\, a, a \,\mathfrak{R}\, f \text{ for some maximal idempotents } e, f\} \cup 0.$$

Then $\mathfrak{I}(S)$ is a primitive inverse semigroup. It follows from the proof that we can always suppose that $w_\alpha = 0$ for some $\alpha \in A$. Call S *balanced* if any two maximal idempotents of S are $\mathfrak{D}$-equivalent.

THEOREM 2. *The following conditions on a 0-simple ω-regular semigroup S are equivalent.*

(i) *S is balanced;*

(ii) *S admits a representation as in Theorem* 1 *with* $w_\alpha = 0$ *for all* $\alpha \in A$;

(iii) $\mathfrak{J}(S)$ *is a Brandt semigroup;*

(iv) *S is isomorphic to a Rees matrix semigroup* $\mathfrak{M}^0(K; A, A; \Delta)$ *over a simple inverse* ω*-semigroup* K, Δ *is the identity matrix.*

The structure of the semigroup K in Theorem 2 was determined by Kočin [1] and Munn [5], the Rees matrix semigroups over bisimple inverse semigroups were studied in [3] (for the 0-simple case in the theorem, cf. [3, Corollary 5.7] and [6, Theorem 4.2]). Various other special cases include: 0-bisimple, combinatorial, balanced, and combinations thereof.

4. For the remaining cases, we will need the following.

Construction III. Let Y be a tree semilattice satisfying one of the two conditions: (1) Y has a zero ζ and all elements of Y are of finite height, (2) Y has no zero and is of locally finite length. To every nonzero element α of Y associate a Brandt semigroup S_α, suppose that the family $\{S_\alpha\}$ is pairwise disjoint, and that a homomorphism $\phi_\alpha: S_\alpha \rightarrow S_{\bar{\alpha}}$ is given, where $\bar{\alpha}$ is the unique element of Y covered by α, with the properties:

(i) $S_\alpha\phi_\alpha \cap S_\beta\phi_\beta = 0$ if $\bar{\alpha} = \bar{\beta}$;

(ii) for every infinite ascending chain $\alpha_1 < \alpha_2 < \cdots$ in Y and every $a \in S^*_{\alpha_1}$, there exists α_k such that $a \notin S_{\alpha_k}\phi_{\alpha_k}\phi_{\alpha_{k-1}} \cdots \phi_{\alpha_2}$. Let $\psi_{\alpha,\alpha}$ be the identity mapping on S_α, and for $\alpha > \beta$, let $\psi_{\alpha,\beta} = \phi_\alpha\phi_{\alpha_1} \cdots \phi_{\alpha_n}$ where $\alpha \succ \alpha_1 \succ \cdots \succ \alpha_n \succ \beta$. Let $S = (\bigcup_{\alpha \in Y \setminus \zeta} (S_\alpha \setminus 0_\alpha)) \cup 0$ where ζ is the zero of Y (if Y has one), and 0 is an element not contained in any S_α, and on S define the multiplication $*$ by

$$a * b = (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta}) \quad \text{if } \alpha\beta \neq \zeta \quad \text{and} \quad (a\psi_{\alpha,\alpha\beta})(b\psi_{\beta,\alpha\beta}) \neq 0_{\alpha\beta} \quad \text{in } S_{\alpha\beta},$$

and all other products are equal to 0. The set S with this multiplication will be called a *Brandt tree* if Y has a zero and a *rooted Brandt tree* otherwise.

THEOREM 3. *A semigroup S is prime* ω*-regular and has a proper* 0*-minimal ideal if and only if S is an ideal extension of a* 0*-simple* ω*-regular semigroup I by a Brandt tree T determined by a* 0*-restricted homomorphism of T into the top of I.*

Such a homomorphism is completely determined by its restriction to the socle $\mathfrak{S}(T)$ of T, so all such homomorphisms are given by 0-restricted homomorphisms of $\mathfrak{S}(T)$ into $\mathfrak{J}(I)$, both of which are primitive inverse semigroups, and are easy to find explicitly.

THEOREM 4. *A groupoid S is a prime ω-regular semigroup without 0-minimal ideals if and only if S is a rooted Brandt tree.*

5. The semigroups $\mathcal{O}(A, w; V, \sigma)$ and $\mathcal{O}[A, w; V, \sigma]$ do not seem to admit a neat isomorphism theorem except in special cases. In the balanced case, using Theorem 2, [**3**, 4.1], and [**1**, Theorem 4], we derive a satisfactory isomorphism theorem. A direct proof does the same in the case these semigroups are combinatorial. Isomorphisms of the semigroups in Construction III are similar to those in [**4**, Théorème 3.1], while isomorphisms of the semigroups in Theorem 3 can be expressed by isomorphisms of I and T satisfying a commutative diagram.

REFERENCES

1. B. P. Kočin, *The structure of inverse ideally-simple ω-semigroups*, Vestnik Leningrad. Univ. **23** (1968), no. 7, 41–50. (Russian) MR **37** #2881.

2. G. Lallement and M. Petrich, *Décompositions I-matricielles d'un demi-groupe*, J. Math. Pures Appl. (9) **45** (1966), 67–117. MR **33** #5765.

3. ———, *A generalization of the Rees theorem in semigroups*, Acta Sci. Math. (Szeged) **30** (1969), 113–132. MR **39** #2899.

4. ———, *Structure d'une classe de demi-groupes réguliers*, J. Math. Pures Appl. **48** (1969), 345–397.

5. W. D. Munn, *Regular ω-semigroups*, Glasgow Math. J. **9** (1968), 46–66. MR **37** #5316.

6. ———, *0-bisimple inverse semigroups*, J. Algebra **15** (1970), 570–588.

7. D. Rees, *On semi-groups*, Proc. Cambridge Philos. Soc. **36** (1940), 387–400. MR **2**,127.

8. N. R. Reilly, *Bisimple ω-semigroups*, Proc. Glasgow Math. Assoc. **7** (1966), 160–167. MR **32** #7665.

THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802