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SETS OF INTERPOLATION FOR MULTIPLIERS

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Let T denote the circle and I a closed ideal of $L^1(T)$ under convolution. Let $\mathfrak{F}I$ denote the set of sequences of complex numbers which are Fourier transforms of elements of I.

$$\mathfrak{F}I = \left\{(\xi_n): \exists f \in I, \ \hat{f}(n) = \xi_n\right\}.$$

A subset E of the integers is called a set of *interpolation* for the multipliers of $\mathfrak{F}I$ $(=M(\mathfrak{F}I))$ if every bounded complex sequence defined on E is the restriction to E of a multiplier of $\mathfrak{F}I$. E is called a *Sidon* set if every bounded complex sequence on E is the restriction to E of the Fourier transform of some measure on T. Answering a question of Y. Meyer we show here that every set of interpolation $E \subseteq Z^+$ for $M(\mathfrak{F}H^1(T))$ is a Sidon set.

Let $A(T)$ denote the Banach space of all analytic continuous functions on T equipped with the supremum norm. Let $\beta = H^1(T) \hat{\otimes} C(T)$ be the Banach space of all elements of $A(T)$ which can be expressed in the form $\sum_1^\infty f_k * g_k$ where $f_k \in H^1(T)$, $g_k \in C(T)$ and such that $\sum_1^\infty \|f_k\|_1 \|g_k\|_\infty < \infty$. The norm $\|\cdot\|_\beta$ in β is the infimum over all such representations. Meyer [1] has shown that the dual of β is precisely $M(\mathfrak{F}H^1(T))$.

THEOREM 1. *β is isometrically isomorphic to $A(T)$.*

PROOF. It is clear that the natural embedding of β in $A(T)$ is norm decreasing. Let $P(\theta) = \sum_1^\nu a_k \exp[in_k\theta]$ be an arbitrary analytic trigonometric polynomial and write $e^{iM\theta}P(\theta)$ as

$$\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \exp[i(n + N)\theta] * \sum_{k=1}^{\nu} b_k \exp[i(n_k + M)\theta]$$

where $b_k = a_k\{1 - |n_k + M - N|/N\}^{-1}$. Choose $M = N - [N^{1/2}]$ and N larger than n_ν. It is clear that as $N \to \infty$, $b_k \to a_k$ for each k. Since the polynomial on the left-hand side is just a translate of the usual Fejer kernel, it has L^1 norm equal to 1. By the choice of M, the sup norm of the polynomial on the right-hand side tends to $\|P(\theta)\|_\infty$ as $N \to \infty$. Hence

$$\|\exp[iM\theta]P(\theta)\|_\beta < \|P(\theta)\|_\infty + \epsilon$$

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for N sufficiently large.

It is clear that $\|P(\theta)\|_\beta \leqq \|\exp[iM\theta]P(\theta)\|_\beta$ for all positive integers M. Hence $\|P(\theta)\|_\beta \leqq \|P(\theta)\|_\infty$. Since the analytic trigonometric polynomials are dense in β the theorem follows.

The following answers a question raised in [**1**, p. 554].

COROLLARY. *$E \subset Z^+$ is a set of interpolation for $M(\mathfrak{F}H^1(T))$ if and only if E is a Sidon set.*

PROOF. The only implication of interest is the "only if" one. Thus assume E is a set of interpolation for $M(\mathfrak{F}H^1(T))$. It is an easy consequence of the definition that E is a set of interpolation for $M(\mathfrak{F}H^1(T))$ if and only if the elements of β whose spectra are contained in E have absolutely convergent Fourier series. Hence there is some constant c, depending only on E, such that

$$\sum_{k=1}^{\nu} |a_k| \leqq c\|P(\theta)\|_\beta$$

for all trigonometric polynomials $P(\theta) = \sum_{k=1}^{\nu} a_k \exp[in_k\theta]$ with spectrum contained in E. Since $\|P(\theta)\|_\infty = \|P(\theta)\|_\beta$, E is a Sidon set (cf. [**3**, p. 121]). Q.E.D.

It is of some interest to compare the above notions of interpolation in $M(\mathfrak{F}I)$ with the following definition implicit in [**1**]: E is said to be a set of *E-interpolation* if every bounded complex sequence on E is the restriction to E of a multiplier of $\mathfrak{F}I(E)$ where $I(E)$ is the ideal of all L^1 functions whose spectrum is contained in E. The concept of Sidon set is replaced here by that of $\Lambda(2)$ set. Recall that E is a $\Lambda(2)$ set if every L^1 function whose spectrum is contained in E is in L^2.

THEOREM 2. *E is a set of E-interpolation if and only if it is a $\Lambda(2)$ set.*

PROOF. The fact that $\Lambda(2)$ sets are sets of E-interpolation is an immediate consequence of the Riesz-Fisher theorem.

Conversely if E is a set of E-interpolation and $P(\theta) = \sum a_k \exp[in_k\theta]$ is an E-polynomial define $g(t, \theta) = \sum a_k\varphi_k(t) \exp[in_k\theta]$ where φ_k is the kth Rademacher function. Then $g_t = \hat{s}_t * f$ where s_t is the convolution operator from $I(E)$ to $L^1(T)$ such that $s_t(\pi_k) = \varphi_k(t)$.

Let $l_{\infty,E}$ denote the quotient space of l_∞ by the closed subspace of those sequences vanishing on E. Then since E is a set of E-interpolation, the natural map $\sigma: M(\mathfrak{F}I(E)) \to l_{\infty,E}$ is onto, and hence has a bounded inverse. Thus $\|s_t\| \leqq c$ where c is independent of t, and $\|g_t\|_1 \leqq c\|f\|_1$. The proof now proceeds as in Theorem 3.1 of [**2**].

Integrate $(\sum |a_k|^2)^{1/2} \leqq 2\int_0^1 |g(t, \theta)|\, dt$ with respect to θ over $[-\pi, \pi]$ and use the above inequality.

REMARK. In direct analogy to the space β, $\beta_E = I(E)\hat{\otimes} C(T)$ may be formed. It may be of interest to ask for what sets E it is true that whenever $F \subset E$, F is a set of $M(\mathfrak{F}I(E))$ interpolation if and only if it is Sidon. By Theorem 2 this will fail if $E = F$ and E is taken to be a set which is $\Lambda(2)$ but not Sidon.

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