QUOTIENTS OF FINITE $W^*$-ALGEBRAS

BY JØRGEN VESTERSTRØM

Communicated by C. C. Moore, May 18, 1970

1. In this note we present results concerning the following problem. Suppose $M$ is a $W^*$-algebra and $J \subseteq M$ a uniformly closed two-sided ideal. Then the quotient algebra $M/J$ is a $C^*$-algebra, and the problem is: What are the conditions that $M/J$ be a $W^*$-algebra?

Since we can write $M$ as a direct sum of a finite and a properly infinite $W^*$-algebra, we can discuss the two cases separately. In [3] and [4] Takemoto solved the problem for a properly infinite $W^*$-algebra, that can be represented on a separable space. His theorem states that $M/J$ is a $W^*$-algebra, if and only if $J$ is ultra-weakly closed.

2. If $M$ is finite the situation is quite different. There are “many” non-ultra-weakly closed ideals $J$ for which the quotient $M/J$ is a $W^*$-algebra. Indeed, Wright [5] and Feldman [1] proved that if $J$ is a maximal ideal, $M/J$ is a finite factor. This result was proved by a different method by Sakai in [2]. The following theorem generalizes that result.

**Theorem 1.** Let $M$ be a finite and $\sigma$-finite $W^*$-algebra with center $Z$. Let $J$ be a uniformly closed two-sided ideal satisfying the following conditions:

(i) $J$ is an intersection of maximal ideals,

(ii) $Z/Z \cap J$ is a $W^*$-algebra,

(iii) $Z/Z \cap J$ is $\sigma$-finite.

Then $M/J$ is a $W^*$-algebra.

As a partial converse we have

**Theorem 2.** If $J$ is a uniformly closed two-sided ideal of the finite and $\sigma$-finite $W^*$-algebra $M$ and $M/J$ is a $W^*$-algebra, then $J$ satisfies the conditions (i) and (ii) of Theorem 1.

**Remark.** If we assume that $M$ can be represented on a separable
space, and if we further assume the continuum hypothesis, the condition (iii) becomes necessary for \( M/J \) to be a \( W^* \)-algebra. Thus, under these conditions, (i), (ii), and (iii) are necessary and sufficient for \( M/J \) to be a \( W^* \)-algebra.

3. Outline of proofs. The necessity. If \( M/J \) is a \( W^* \)-algebra, it is necessarily a finite \( W^* \)-algebra. Since the intersection of maximal ideals in the finite \( W^* \)-algebra \( M/J \) is \( \{0\} \), \( J \) is an intersection of maximal ideals. Moreover, \( Z/Z \cap J \) is isomorphic to the center of \( M/J \), and is therefore a \( W^* \)-algebra. Under the continuum hypothesis, a simple cardinality argument shows, that \( Z/Z \cap J \) must be \( \sigma \)-finite, if we also assume that \( M \) can be represented on a separable space.

The sufficiency. Suppose that (i), (ii), and (iii) are satisfied. We consider the Banach \( Z \)-module \( \mathcal{U} \) generated by the maps \( x \in M \to (ax)\# \in Z \), where \( a \in M \). \( \# \) is the canonical center valued trace on \( M \). Since \( J \) is an intersection of maximal ideals, it is invariant under \( \mathcal{U} \), and we can factor \( \mathcal{U} \) to \( \tilde{\mathcal{U}} \), consisting of linear maps \( M/J \to Z/Z \cap J \). By conditions (ii) and (iii) there is a normal faithful state \( \phi \) on \( Z/Z \cap J \). Let \( F \) be the set of all linear functionals of the form \( \mu \circ \Phi \), where \( \Phi \in \tilde{\mathcal{U}} \). By Sakai's criterion it suffices to prove that \( M/J \) is a dual space of \( F \). Let \( E \) be the completion of \( F \). By a theorem in [2], it suffices to prove that for every \( f \in E \), there is an \( x \in M/J \) with \( \|x\| = 1 \) and \( f(x) = \|f\| \). By means of polar decomposition for elements of \( \tilde{\mathcal{U}} \) this is easily proved, if \( f \in F \). In order to extend this result to \( E \), we decompose the functionals in \( E \) over \( Z/Z \cap J \), and by applying a technique similar to that of standard measure theory, we obtain the desired result for \( f \in E \).

4. Thus the problem of finding the \( W^* \)-quotients of the finite \( W^* \)-algebra \( M \) (which is supposed to act on a separable space), is reduced to the corresponding problem for the center \( Z \). Since we assume that \( M \) acts on a separable space we may assume that \( Z = L^m_\infty (I) \), the space of essentially bounded Lebesgue measurable functions on the unit interval, or \( Z = l^\infty (N) \), the set of bounded complex sequences.

Theorem 3. There exist countably many surjective nonnormal *-homomorphisms \( \Phi : Z_1 \to Z_2 \) with pairwise different kernels in each of the following three cases:

(i) \( Z_1 = L^m_\infty (I) \), \( Z_2 = L^m_\infty (I) \),

(ii) \( Z_1 = L^m_\infty (I) \), \( Z_2 = l^\infty (N) \),

(iii) \( Z_1 = l^\infty (N) \), \( Z_2 = l^\infty (N) \).

Thus we see that in general there are many non-ultra-weakly closed
ideals $I$ for which $Z/I$ is a $W^*$-algebra. To give nontrivial examples in the noncommutative case we need only consider the finite $W^*$-algebra $F \otimes Z$ where $F$ is a finite factor. The center of $F \otimes Z$ is $Z$, and if $I \subseteq Z$ is an ideal, let $J$ be the ideal of $M$ which is the intersection of the maximal ideals that contain $I$. If $Z/I$ is a $\sigma$-finite $W^*$-algebra, then so is $F \otimes Z/J$, and $J$ is ultra-weakly closed if and only if $I$ is.

5. As to the existence of non $W^*$-quotients we have the following theorem:

**Theorem 4.** Let $Z$ be an infinite dimensional abelian $W^*$-algebra, which can be represented on a separable space. Then $Z$ admits a $C^*$-quotient, which is not a $W^*$-algebra.

**Proof.** If $Z = l^\infty(N)/c_0(N)$ will do. If $Z = L_m^\infty(I)$ we apply Theorem 3, part (ii).

**Theorem 5.** (i) There exists a pair $J$, $M$ such that the center of $M/J$ is a $\sigma$-finite $W^*$-algebra, but $J$ is not an intersection of maximal ideals.

(ii) There exists a pair $J$, $M$ such that $J$ is an intersection of maximal ideals, but the center of $(M/J)$ is not a $W^*$-algebra.

In the proof of Theorem 5 (ii) we apply Theorem 4. To prove Theorem 5 (i) we need the following considerations. Let $M$ be a finite $W^*$-algebra. The maximal ideal space $\text{Max}(M)$ considered as a subset of the primitive ideal space $\text{Prim}(M)$ is homeomorphic to $\text{Max}(Z)$. Let $x(J)$ be the image of $x \in M$ in $M/J$ under $M \rightarrow M/J$, if $J \subseteq \text{Max}(M)$. Then the following is applicable in the proof of Theorem 5 (i).

**Theorem 6.** Let $J_0 \in \text{Max}(M)$. Then the following two conditions are equivalent:

(i) There is no primitive ideal $J$ such that $J \neq J_0$ and $J \cap Z = J_0 \cap Z$.

(ii) The functions $J \in \text{Max}(M) \rightarrow \|x(J)\|$, $x \in M$, are continuous at $J_0$.

The author is indebted to C. Akemann, who pointed out the example of Theorem 3 (iii). The author also wants to express his gratitude to M. Takesaki, who directed the author's attention to these problems, and to S. Sakai and E. Stormer for valuable conversations and advice.

Detailed proofs will appear elsewhere.

**References**


MATEMATISK INSTITUT, AARHUS UNIVERSITET, 8000 AARHUS C., DENMARK