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ENDOMORPHISMS OF EXACT SEQUENCES[1]

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Let E and F denote the following exact sequences

$$E\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \qquad F\colon 0 \to A \xrightarrow{f} B \xrightarrow{hg} D \xrightarrow{j} E \to 0$$

where hg represents the canonical factorization of the middle morphism of F into an epimorphism g followed by a monomorphism h. We shall take the term "endomorphism" of E or F to mean a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \\ & 1\| & & \downarrow\alpha & & \|1 & \\ 0 \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to 0 \end{array} \quad \text{or} \quad \begin{array}{ccccccccc} 0 \to & A & \xrightarrow{f} & B & \xrightarrow{hg} & D & \xrightarrow{j} & E & \to 0 \\ & 1\| & & \downarrow\beta & & \downarrow\gamma & & \|1 & \\ 0 \to & A & \xrightarrow{f} & B & \xrightarrow{hg} & D & \xrightarrow{j} & E & \to 0. \end{array}$$

We shall compute the "endomorphism groups" of E and F and prove that $\mathrm{Aut}(E) = \mathrm{End}(E) \cong \mathrm{Hom}(C, A)$ and that $\mathrm{End}(F) \cong \mathrm{Hom}(h, g)$ where the second Hom is a functor on a category of morphisms with range the category of semigroups.

1. **Notation.** Let R denote a fixed ring with unit and $\mathfrak{M}$ the category of left R-modules. Let $\mathcal{E}$ denote the category of all short-exact sequences E, which begin with A and end with C, and whose morphisms are all triples $(1, \theta, 1) = \theta^\#$ which induce commutative diagrams

$$\begin{array}{cccccccc} E\colon & 0 \to & A & \to & B & \to & C & \to 0 \\ \theta^\# \downarrow & & \|1 & & \downarrow\theta & & \|1 & \\ E'\colon & 0 \to & A & \to & B' & \to & C & \to 0. \end{array}$$

By the 5-lemma, θ is an isomorphism, and thus $\theta^\#$ is one too. Thus every endomorphism of E is an "automorphism."

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The category $\mathfrak{F}$ of all sequences of length two beginning with A and ending at E can be similarly defined. Not all morphisms of $\mathfrak{F}$ need be isomorphisms however.

Let $\mathfrak{M}^2$ denote the abelian category whose objects are all the morphisms of $\mathfrak{M}$, and whose morphisms are all pairs $\binom{\rho}{\sigma}:h\to g$ which give rise to commutative squares

$$\begin{array}{ccc} \cdot & \xrightarrow{\ \rho\ } & \cdot \\ h\downarrow & & \downarrow g \\ \cdot & \xrightarrow[\ \sigma\ ]{} & \cdot \end{array} .$$

One should note that there is no way of adding endomorphisms in $\mathcal{E}$ and $\mathfrak{F}$; one may only compose them with each other.

2. **Computation of** $\mathrm{Aut}_{\mathcal{E}}(\boldsymbol{E})$.

THEOREM 1. $\mathrm{Aut}_{\mathcal{E}}(\boldsymbol{E})\cong\mathrm{Hom}_{\mathfrak{M}}(C, A)$.

PROOF. Let $\alpha^{\#}=(1, \alpha, 1):\boldsymbol{E}\to\boldsymbol{E}$. Set $\alpha-1=\lambda$ or $\alpha=1+\lambda$. Since $\alpha f=f$, $(1+\lambda)f=f+\lambda f=f$, so $\lambda f=0$. Therefore there is a unique morphism $\mu:C\to B$ such that $\lambda=\mu g$; so $\alpha=1+\mu g$. But $g\alpha=g$ implies $g(1+\mu g)=g+g\mu g=g$, so $g\mu g=0$. But g is an epimorphism, so $g\mu=0$. Therefore there is a unique morphism $\nu:C\to A$ such that $\mu=f\nu$. It follows that $\alpha=1+f\nu g$. Moreover, ν is unique because if $\alpha=1+f\nu g=1+f\nu''g$ then $f\nu g=f\nu''g$; but f is a monomorphism, so $\nu g=\nu''g$; similarly g is an epimorphism, so $\nu=\nu''$.

This construction produces a unique mapping

$$\Phi\colon \mathrm{Aut}_{\mathcal{E}}(\boldsymbol{E}) \to \mathrm{Hom}(C, A)$$

where $\Phi(\alpha^{\#})=\nu$. The former is a multiplicative group, the latter an additive abelian group. We must prove that Φ is an isomorphism of groups.

If $\rho:C\to A$ is any morphism of $\mathrm{Hom}(C, A)$, then $(1+f\rho g)^{\#}=(1, 1+f\rho g, 1)$ is an automorphism of $\boldsymbol{E}$ because it clearly gives rise to an appropriate commutative diagram. Hence it follows that $\Phi((1+f\rho g)^{\#})=\rho$, so Φ is onto.

If $\Phi(\alpha^{\#})=\Phi(\alpha_1^{\#})=\nu$, then $\alpha=1+f\nu g=\alpha_1$, so $\alpha_1^{\#}=\alpha^{\#}$. Therefore Φ must be a one-to-one mapping.

If $\alpha=1$ then $1^{\#}$ is the identity morphism on $\boldsymbol{E}$ in $\mathcal{E}$, and $\Phi(1^{\#})=0$. Certainly $\alpha^{-1}=1-f\nu g$.

Notice that $(1+f\eta g)(1+f\rho g)=1+f\eta g+f\rho g=1+f(\eta+\rho)g$. Therefore, if $\alpha=1+f\eta g$ and $\alpha'=1+f\rho g$, then $\Phi(\alpha^{\#}\alpha'^{\#})=\eta+\rho=\Phi(\alpha^{\#})$

$+\Phi(\alpha'^{\#})$. Therefore Φ is a homomorphism of groups, and in fact an isomorphism.

It is interesting to note that these automorphism groups are abelian, and are independent of the extension class of $\boldsymbol{E}$.

COROLLARY. $\mathrm{Aut}_{\mathcal{E}}(E)$ *is isomorphic to a (commutative) subgroup of* $\mathrm{Aut}_{\mathfrak{M}}(B)$.

There is a subgroup of $\mathrm{Aut}_{\mathfrak{M}}(B)$ of this type for each submodule A' of B, where the short exact sequence would be $0\to A'\to B\to B/A'\to 0$. If $\mathrm{Aut}_{\mathfrak{M}}(B)$ is the direct limit of the $\mathrm{Aut}_{\mathcal{E}}(E)$ groups, then it would have to be abelian.

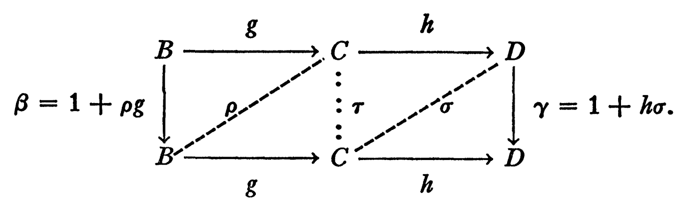
In a following paper, Theorem 1 will be used to prove that the Whitehead group of $\mathcal{E}$ is $\mathrm{Hom}(C, A)$.

3. **Computation of** $\mathrm{End}_{\mathfrak{F}}(F)$. By the same argument as was used in Theorem 1, if $(1, \beta, \gamma, 1): F\to F$ is an endomorphism of F, then there is a unique morphism $\rho: C\to B$ such that $\beta = 1+\rho g$ and a unique morphism $\sigma: D\to C$ such that $\gamma = 1+h\sigma$.

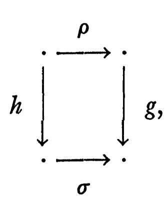
There are two endomorphisms $C\to C$ produced by this, $g\rho$ and σh. One has

$$h(1+\sigma h)g = hg + h\sigma hg = (1+h\sigma)hg = \gamma hg$$
$$= hg\beta = hg(1+\rho g) = hg + hg\rho g.$$

Therefore $hg\rho g = h\sigma hg$, and since g is an epimorphism and h is a monomorphism, $g\rho = \sigma h$.



Therefore there is induced a unique morphism $\tau = 1+g\rho = 1+\sigma h$: $C\to C$ such that $\tau g = g\beta$ and $h\tau = \gamma h$. Also, there is a commutative diagram



$\begin{pmatrix}\rho\\ \sigma\end{pmatrix}: h\to g,$

corresponding to a morphism in the category $\mathfrak{M}^2$.

If we define a multiplication on $\mathrm{Hom}_{\mathfrak{M}^2}(h, g)$ by setting

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix}\begin{pmatrix}\rho'\\ \sigma'\end{pmatrix}=\begin{pmatrix}\rho+\rho'+\rho g\rho'\\ \sigma+\sigma'+\sigma h\sigma'\end{pmatrix},$$

then it is easily checked that this is associative and that $\binom{0}{0}$ is a unit with respect to this multiplication. Therefore, using the same techniques as in Theorem 1, one proves

THEOREM 2. *There is an isomorphism of semigroups*

$$\mathrm{End}_{\mathfrak{F}}(F) \cong \mathrm{Hom}_{\mathfrak{M}^2}(h, g).$$

It would be very interesting if one could give some way of deciding which endomorphisms are automorphisms. An equivalent problem is to determine when $g\rho$ is quasi-regular in $\mathrm{Aut}_{\mathfrak{M}}(C)$.

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