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ON REGULARITY OF GENERALIZED SURFACES OF CONSTANT MEAN CURVATURE

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Statement of result. A surface in a manifold is called "regular" to emphasize that it is immersed; the term "generalized surface" is used when regularity is only assumed almost everywhere.

We treat generalized surfaces of constant mean curvature in an analytic Riemannian manifold M of dimension 3. To this end consider the variational problem:

$$E[X] = D[X] + 4HV[X] \to \min$$

where the admissible X are all mappings of a domain $G \subset R^2$ into M, continuous in $\overline{G}$ and C^2 in G, which agree with a given admissible X_0 on ∂G up to reparameterization; and where H is a real constant, $D[X]$ is the Dirichlet integral of X

$$D[X] = \iint_G g_{kl}(\overset{k}{x}_u \overset{l}{x}_u + \overset{k}{x}_v \overset{l}{x}_v)\, du\, dv$$

and $V[X]$ is volume of a consistently chosen region whose oriented boundary is $X - X_0$. A mapping which is stationary for this functional is a generalized surface of constant mean curvature H in isothermal coordinates, cf. [1, pp. 107–112]. If a mapping is not only stationary but minimizing, we can say more:

THEOREM. *Any mapping X which minimizes E among the admissible mappings defines a regular surface.*

Observe that the mapping X itself need not be a regular parameterization. Since the theorem can be applied in any neighborhood, it follows that a mapping which minimizes E even locally must be regular. The theorem is a generalization of the work of Osserman on the Plateau problem, that is, the case $H=0$, $M=E^3$ [**3**].

Consequences. In the case that M is a space form, examples can be constructed of generalized surfaces of constant mean curvature with

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singularities resembling branch points of conformal mappings. This construction was given by Courant for the case $H=0$, $M=E^3$ [**1**, pp. 122ff]. By the above theorem these generalized surfaces are unstable with respect to E in every neighborhood of the singular point, in contrast to what is known for single-integral variational problems.

Combining the above theorem with various results on the existence of generalized surfaces as solutions to variational problems of this type improves each to a statement of existence for regular surfaces. For example, the result of Hildebrandt [**2**] now yields

THEOREM. *Given a continuous curve γ in the unit ball of E^3 such that some mapping with finite Dirichlet integral spans γ, and given H with $|H| \leqq 1$, there exists a regular surface spanning γ with constant mean curvature H.*

This answers a question posed in [**2**], and, in similar fashion, the conjecture in [**3**, §5.4] for the analytic case.

Indication of proof. The approach taken is to study the behavior of a solution to the Euler equations near a branch point, i.e. a point of nonregularity. Orthogonal projection on a certain plane is shown to be a covering map in a punctured neighborhood. Properties of nonparametric solutions are then applied toward an analysis of the self-intersections of the solution. We prove

THEOREM. *There is an analytic arc of self-intersection of the solution which converges to the branch point in a well-defined direction.*

LEMMA. *There is a neighborhood of the branch point in which either*

(1) *the solution intersects itself only transversally*; *or*

(2) *with appropriate choice of parameters, the solution is invariant under a rotation of the parameter domain.*

The main result is then obtained in case (1) by means of a discontinuous "reparameterization" near the branch point which leaves the value of E unchanged but introduces a corner. This new generalized surface cannot minimize E, nor, therefore, can the original solution. In case (2) the solution has an inessential branch point; the surface itself is regular.

This work can be generalized to a broad class of geometrically meaningful variational problems whose Euler equations form elliptic systems. In particular, geometric side conditions may be imposed.

Details will appear in the author's thesis.

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