

## CHEVALLEY GROUPS OVER COMMUTATIVE RINGS<sup>1</sup>

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**1. Introduction.** Steinberg [8] has given a simple presentation for the universal central extension [7], [8], [9] of the group of rational points of a simply connected Chevalley group over a field. In this note we announce a similar theory for the simply connected Chevalley groups over *commutative rings* and outline the proof of a stability theorem for certain functors resulting from this construction. Complete proofs will appear elsewhere.

Let us introduce some notation.  $A$  denotes a commutative ring with 1,  $A^*$  is its group of invertible elements,  $\mathfrak{p}$  and  $\mathfrak{q}$  are ideals of  $A$ , and  $(1 + \mathfrak{q})^* = (1 + \mathfrak{q}) \cap A^*$ .  $\Phi$  is a reduced irreducible root system [2] and  $G(\Phi, \ )$  is the simply connected Chevalley-Demazure group scheme with root system  $\Phi$ . If  $\Phi$  is of type  $C_l$ ,  $l \geq 1$  ( $C_1 = A_1$ ), we say  $\Phi$  is *symplectic*, and if  $\Phi$  is of type  $A_l$ ,  $B_l$ ,  $C_l$ , or  $D_l$ , we say  $\Phi$  is *classical*. The subgroup of  $G(\Phi, A)$  generated by the elementary unipotents  $e_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in A$ , will be denoted  $E(\Phi, A)$ . A full discussion of these notions may be found in [3], [5], and [9].

Define the Steinberg group,  $\text{St}(\Phi, A)$ , to be the group with generators  $x_\alpha(t)$ ,  $\alpha \in \Phi$ ,  $t \in A$ , subject to the relations

$$(1.1) \quad \begin{aligned} x_\alpha(s)x_\alpha(t) &= x_\alpha(s+t) & (\alpha \in \Phi; s, t \in A) \\ [x_\alpha(s), x_\beta(t)] &= \prod x_{i\alpha+j\beta}(N_{\alpha,\beta,i,j}s^i t^j) & (\alpha, \beta \in \Phi, \alpha + \beta \neq 0) \end{aligned}$$

where the product is as in [8]. Since the elementary unipotents  $e_\alpha(t)$  also satisfy these relations, the map  $x_\alpha(t) \mapsto e_\alpha(t)$  extends to a homomorphism  $\pi: \text{St}(\Phi, A) \rightarrow G(\Phi, A)$  with image  $E(\Phi, A)$ . Set  $\ker \pi = L(\Phi, A)$ .

In §2 we present certain commutator formulas which yield necessary and sufficient conditions for  $E(\Phi, A)$  and  $\text{St}(\Phi, A)$  to be their own derived groups. In §3 we show that the extension  $\text{St}(\Phi, A) \rightarrow E(\Phi, A)$

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is “stably central,” and, under certain restrictions on  $\text{rk } \Phi$ , that every central extension of  $\text{St}(\Phi, A)$  splits. This generalizes results of Milnor [6], Kervaire [4] and Steinberg [9] for  $\text{SL}(n, A)$ ,  $n \geq 3$  (the Chevalley groups of type  $A_n$ ,  $n \geq 2$ ). The functor  $\lim_{n \rightarrow \infty} L(A_n, \ )$  is Milnor’s algebraic  $K_2$  functor, and similar functors arise from the results of §2 and §3 for the other Chevalley groups.

Now suppose  $\Phi_l$  is a root system of rank  $l$ . An inclusion  $\Phi_l \subset \Phi_{l+1}$  induces homomorphisms of the corresponding groups and, in particular, a map  $\theta_l: L(\Phi_l, A) \rightarrow L(\Phi_{l+1}, A)$ . When  $A$  is a field, Steinberg [8] has shown that the maps  $\theta_l$  are always surjective. This partially answers a stability question: how large must  $l$  be, relative to the dimension of the maximal ideal space of  $A$ , for  $\theta_l$  to be surjective? In §4 we show that  $l=1$  suffices for almost all semilocal rings—rings with  $\dim \max A$  equal to 0. Consequences of this result will be discussed in §4.

2. Commutators in Chevalley groups. Set

$$\text{St}(\Phi, q) = \ker(\text{St}(\Phi, A) \rightarrow \text{St}(\Phi, A/q)).$$

The image of  $\text{St}(\Phi, q)$  under  $\pi$  is denoted  $E(\Phi, q)$ , and  $L(\Phi, q) = \ker \pi \cap \text{St}(\Phi, q)$ . We let  $E(\mathfrak{p}, q)$  be the smallest normal subgroup of  $E(\Phi, A)$  containing  $\{e_\alpha(p) \mid p \in \mathfrak{p}, \alpha \text{ short}\} \cup \{e_\beta(q) \mid q \in \mathfrak{q}, \beta \text{ long}\}$ . If  $\Phi$  has only one root length, by convention all roots are long.

(2.1)  $[E(q), E(\mathfrak{p})] \supseteq E((u-1)\mathfrak{p}, (u^n-1)\mathfrak{p})$  for all  $u \in (1+q)^*$ , where  $n=1$  if  $\Phi$  is nonsymplectic and  $n=2$  if  $\Phi$  is symplectic.

(2.2) Let  $d[q]$  be the ideal generated by  $\{q^2-q \mid q \in \mathfrak{q}\}$  and let  $s[q]$  be the ideal generated by  $\{q^2 \mid q \in \mathfrak{q}\}$ . It has been shown by Bass and Tate [1] that  $d[A] = \cap \mathfrak{m}$ , where  $\mathfrak{m}$  ranges over all maximal ideals of  $A$  such that  $A/\mathfrak{m} \approx F_2$ . Moreover,  $d[A/q] \subset d[q] \subset \mathfrak{q}$ .

Let  $F$  be a subgroup of  $G(\Phi, A)$  normalized by  $E(\Phi, A)$  and, set  $F' = [E(q), F]$ .

(2.3) THEOREM. Assume  $\text{rk } \Phi \geq 2$ , and suppose for some  $\gamma \in \Phi$  and some  $q_0 \in A$  that  $e_\gamma(q_0) \in F$ . Then  $E(\mathfrak{a}, \mathfrak{b}) \subset F'$ , where  $\mathfrak{a} = \mathfrak{b} = \mathfrak{q}q_0$ , except in the following cases:

$\gamma \text{ long: } \Phi = G_2$	$\mathfrak{a} = d[q]q_0$
$\Phi = C_2$	$\mathfrak{a} = d[q]q_0 + qd[Aq_0]$
	$\mathfrak{b} = d[q]q_0^2 + qd[Aq_0]q_0 + 2qq_0$
$\Phi = C_l, \quad l > 2$	$\mathfrak{b} = qq_0^2 + 2qq_0 + s[q]q_0$
$\gamma \text{ short: } \Phi = G_2$	$\mathfrak{a} = 2qq_0 + d[q]q_0^3 + 3d[q]q_0$
	$\mathfrak{b} = qq_0^3 + 3qq_0$

$$\begin{aligned} \Phi = C_2 & & \mathfrak{a} &= d[q]q_0 + qd[Aq_0] \\ & & \mathfrak{b} &= d[q]q_0^2 + qd[Aq_0]q_0 + 2qq_0 \\ \Phi = B_l, C_l, F_4, \quad l > 2 & & \mathfrak{b} &= qq_0^2 + 2qq_0. \end{aligned}$$

Moreover, if  $A$  has no residue field with two elements, the case  $\Phi = C_2$  is the same as  $\Phi = C_l, l > 2$  when  $\gamma$  is long.

(2.4) COROLLARY.  $[E(\Phi, A), E(\Phi, q)] = E(\Phi, q)$  provided, when  $\Phi = C_2$  or  $G_2$ , that  $A$  has no residue field with two elements, and when  $\Phi = A_1$ , that the elements  $u^2 - 1, u \in A^*$ , generate the unit ideal of  $A$ .

The case of  $A_1$  follows from (2.1); the others from (2.2) and (2.3), with  $F = E(A), q_0 = 1$ .

REMARKS. (a) For  $q = A$ , the hypotheses of (2.4) are necessary and sufficient. To see this, note that  $E(A) \rightarrow E(A/\mathfrak{p})$  is surjective, which shows that  $E(A/\mathfrak{p})$  is its own derived group whenever  $E(A)$  is. However, it is well known that the groups  $SL(2, F_2), SL(2, F_3), Sp(4, F_2)$ , and  $G_2(F_2)$  contain normal subgroups of index 2.

(b) (2.1), (2.3), and (2.4) remain true if  $E$  is replaced by  $St$  throughout and  $F$  is taken to be a normal subgroup of  $St(\Phi, A)$ .

3. Central extensions and  $H_2$ . Recall [7], [9] that the universal central extension of a group  $G$  is a central extension,  $\hat{G}$ , of  $G$  which is its own derived group and which has no nonsplit central extensions of its own. These conditions characterize  $\hat{G}$  up to unique isomorphism, and  $\ker(\hat{G} \rightarrow G) \approx H_2(G, \mathbf{Z})$ .

(3.1) THEOREM. Let  $\Phi$  be a reduced irreducible root system, of rank  $\geq 5$  if  $\Phi$  is of type  $B_l$  or  $D_l$ , and of rank  $\geq 4$  otherwise. Then  $St(\Phi, A)$  has no nonsplit central extensions. If  $A$  has no residue field with two elements, the same is true for  $\Phi = C_3$  or  $B_4$ . If the elements  $u^2 - 1, u \in A^*$ , generate the unit ideal of  $A$ , then  $St(\Phi, A)$  has no nonsplit central extensions whenever  $\text{rk } \Phi \geq 3$ .

OUTLINE OF PROOF. Given a central extension  $p: F \rightarrow St(\Phi, A)$ , we must construct a section  $s: St(\Phi, A) \rightarrow F$  of  $p$ . Over each subgroup of type  $A_2, B_3$ , and  $C_3$  we define canonical liftings,  $y_\alpha(t)$ , of the generators  $x_\alpha(t)$  of  $St(\Phi, A)$  belonging to that subgroup and then prove that the liftings so defined are independent of the subgroup chosen. Finally we verify that relations (1.1) hold for the elements  $y_\alpha(t)$ , showing that  $x_\alpha(t) \mapsto y_\alpha(t)$  defines a homomorphism which is the desired section for  $p$ . Each step involves technical considerations which differ from root system to root system, accounting for the rather complicated hypotheses of (3.1).

REMARK. For  $SL(n, A)$ , (3.1) is due independently to Steinberg [9]

and Kervaire [4]. It is possible to weaken the hypotheses on  $A$  slightly for the cases  $\Phi = A_3, B_3, D_4$  (e.g.  $D_4(F_3)$  has no nonsplit central extensions). If  $A$  has enough units, the theorem may be extended to groups of rank  $< 3$  using the method of [8].

(3.2) THEOREM. *Let  $\Phi_0$  be a simple system in  $\Phi$  [3], let  $\alpha \in \Phi_0$ , and denote by  $\Phi'$  the subsystem of  $\Phi$  generated by  $\Phi_0 - \{\alpha\}$ . Then  $\ker \pi \cap M$  is central in  $\text{St}(\Phi, A)$ , where  $M$  is the image of  $\text{St}(\Phi', A)$  in  $\text{St}(\Phi, A)$  under the map induced by  $\Phi' \subset \Phi$ .*

COROLLARY. *If  $\Phi, A$  are as in (3.1), then  $\text{St}(\Phi, A) / [\ker \pi, \text{St}(\Phi, A)]$  is the universal central extension of  $E(\Phi, A)$ . In particular, whenever  $\pi$  is central,  $L(\Phi, A) \approx H_2(E(\Phi, A), \mathbf{Z})$ .*

If  $\Phi$  is classical, let  $\text{St}_\infty(\Phi, A), E_\infty(\Phi, A), L_\infty(\Phi, A)$  be the direct limits as  $l \rightarrow \infty$  of the groups  $\text{St}(\Phi_l, A), E(\Phi_l, A), L(\Phi_l, A)$ .

(3.3) COROLLARY. *If  $\Phi$  is classical, then  $\text{St}_\infty(\Phi, A)$  is the universal central extension of  $E_\infty(\Phi, A)$  and  $L_\infty(\Phi, A) \approx H_2(E_\infty(\Phi, A), \mathbf{Z})$ .*

4. **Stability in dimension 0.** For  $u \in A^*, \alpha \in \Phi$ , define elements  $\hat{h}_\alpha(u)$  as in [8]. Let  $\hat{H}(\Phi, A)$  be the subgroup of  $\text{St}(\Phi, A)$  generated by the  $\hat{h}_\alpha(u)$ , and let  $\hat{H}(\Phi, q)$  be the smallest normal subgroup of  $\hat{H}(\Phi, A)$  containing all  $\hat{h}_\alpha(v), v \in (1+q)^*, \alpha \in \Phi$ .

The pairing  $(u, v) \mapsto \{u, v\}_l = \hat{h}_\alpha(uv)\hat{h}_\alpha(u)^{-1}\hat{h}_\alpha(v)^{-1}$  takes values in  $L(\Phi_l, q)$  if  $u$  or  $v$  is in  $(1+q)^*$  and is independent of the long root  $\alpha$  chosen. Denote the subgroup of  $L(\Phi_l, q)$  generated by the values of  $\{ , \}_l$  by  $D(\Phi_l, q)$ .  $D(\Phi_l, q)$  is a central subgroup of  $\text{St}(\Phi_l, A)$  (cf. [9]), and the induced map  $D(\Phi_l, q) \rightarrow D(\Phi_{l+1}, q)$  is clearly surjective.

If  $S$  is a subset of  $A$ , we write  $\mathbf{Z}[S]$  for the subring of  $A$  generated by  $S$ .

(4.1) THEOREM. *Let  $q$  be an ideal contained in  $\text{rad } A$ . If  $\Phi$  is symplectic assume  $A = \mathbf{Z}[(A^*)^2]$ ; otherwise assume  $A = \mathbf{Z}[A^*]$ . Then  $L(\Phi, q) = D(\Phi, q)$ . In particular, the restrictions of the maps  $\theta_l$  of §1 to  $L(\Phi_l, q)$  are surjective.*

(4.2) THEOREM. *Let  $A$  be a semilocal ring with at most one residue field isomorphic to  $F_2$ . If  $\Phi$  is symplectic, assume further that  $A = \mathbf{Z}[(A^*)^2]$  (this is automatic if  $2 \in A^*$ ). Then  $L(\Phi, A) = D(\Phi, A)$  and the maps  $\theta_l$  of §1 are surjective. Moreover, if  $\Phi$  and  $A$  are as in (3.1),  $\text{St}(\Phi, A)$  is the universal central extension of  $E(\Phi, A)$  and  $L(\Phi, A) \approx H_2(E(\Phi, A), \mathbf{Z})$ .*

*Note.* Matsumoto [5] has shown the injectivity of  $\theta_l$  when  $A$  is a field. A paper of the author's now in preparation describes certain

new identities satisfied by  $\{ , \}$  which imply this injective stability theorem for a radical ideal in a semilocal ring generated by its units.

The proof of (4.1) is based on the following decomposition of the group  $\text{St}(\Phi, \mathfrak{q})$  when  $\mathfrak{q} \subset \text{rad } A$ , similar to the Bruhat decomposition [2], [9] of the Chevalley groups.

In  $\text{St}(\Phi, \mathfrak{q})$  let  $\hat{U}(\Phi, \mathfrak{q})$  be the subgroup generated by all  $x_\alpha(q)$ ,  $\alpha > 0$ ,  $q \in \mathfrak{q}$ , and  $\hat{U}^-(\Phi, \mathfrak{q})$  be the subgroup generated by all  $x_\alpha(q)$ ,  $\alpha < 0$ ,  $q \in \mathfrak{q}$ .

(4.3) THEOREM. *The product map*

$$\hat{U}^-(\Phi, \mathfrak{q}) \times \hat{H}(\Phi, \mathfrak{q}) \times \hat{U}(\Phi, \mathfrak{q}) \xrightarrow{\psi} \text{St}(\Phi, \mathfrak{q})$$

is injective, and  $L(\Phi, \mathfrak{q}) \cap \text{im } \psi \subset \hat{H}(\Phi, \mathfrak{q})$ . If  $\psi$  is surjective, then  $\mathfrak{q} \subset \text{rad } A$ .

Conversely, suppose  $\mathfrak{q} \subset \text{rad } A$  and assume  $A = \mathbf{Z}[(A^*)^2]$  (resp.  $\mathbf{Z}[A^*]$ ) if  $\Phi$  is symplectic (resp. nonsymplectic). Then  $\psi$  is surjective.

Using these theorems together with known properties [8], [9] of the pairing  $\{ , \}$ , one may derive quantitative information about the groups  $L(\Phi, A)$  and, in particular, about  $K_2(A)$ . Some examples are:

COROLLARY. *Let  $m \in \mathbf{Z}$ ,  $m > 0$ ,  $m \not\equiv 0 \pmod{4}$ . Then  $L(\Phi, \mathbf{Z}/m\mathbf{Z}) = 0$ .*

For  $K_2$ , this was proved by Milnor [6] using his computation of  $K_2(\mathbf{Z})$  and results of Mennicke, Bass, Lazard, and Serre on the congruence subgroup problem. More generally:

COROLLARY. *Let  $\mathfrak{D}$  be a Dedekind domain of characteristic 0,  $0 \neq \mathfrak{p} \subset \mathfrak{D}$  a prime ideal which is unramified over  $\mathfrak{p}\mathbf{Z} = \mathfrak{p} \cap \mathbf{Z}$ . If  $\text{rk } \Phi = 1$ , assume that  $\mathfrak{D}/\mathfrak{p} \neq F_9$ . Then if  $p$  is odd,  $L(\Phi, \mathfrak{D}/\mathfrak{p}^n) = 0$  for all  $n \geq 1$ . Moreover, if  $\Phi$  is nonsymplectic and  $p = 2$ ,  $L(\Phi, \mathfrak{p}^{n-1}/\mathfrak{p}^n)$  is the product of at most  $2^n - 1$  cyclic groups of order 2, where  $\mathfrak{D}/\mathfrak{p}$  has cardinality  $2^n$ .*

COROLLARY. *The map  $H_2(\text{SL}(2, \mathbf{Z}/2^n\mathbf{Z}), \mathbf{Z}) \rightarrow L(A_1, \mathbf{Z}/2^n\mathbf{Z})$  is surjective for  $n = 1, 2$ , but not for  $n \geq 3$ .*

Note. This corollary implies that  $\{-1, -1\} \neq 0$  in  $L(A_1, \mathbf{Z}/4\mathbf{Z})$  (cf. [6]). This does not imply a similar result for  $K_2(\mathbf{Z}/4\mathbf{Z})$ , since it is only known that the map  $L(A_1, \mathbf{Z}/4\mathbf{Z}) \rightarrow K_2(\mathbf{Z}/4\mathbf{Z})$  is surjective.

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