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A NEW PROOF OF THE EXISTENCE OF A TRACE IN A FINITE VON NEUMANN ALGEBRA

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The construction of the dimension function for projections in the various types of factor, and the definition of the trace in a factor of type II_1 first appeared in a classical paper of Murray and von Neumann [7]. The proof of the additivity and weak continuity of the trace appeared in [8]. Subsequent authors [2], [5], [6] have demonstrated the existence of traces on a larger class of von Neumann algebras, but all have employed some variant of the Murray-von Neumann method of proof. The purpose of the present paper is to provide a short and independent proof of the following theorem.

THEOREM. *Let $\mathfrak{R}$ be a finite von Neumann algebra, with centre $\mathfrak{C}$, and let $\mathfrak{U}$ be the group of unitary elements of $\mathfrak{R}$.*

(1) *If h is an ultraweakly continuous linear form on $\mathfrak{C}$, then there is a unique linear form g on $\mathfrak{R}$ such that*

(i) *g is ultraweakly continuous,*

(ii) *$g(A) = g(U^*AU)$ for $A \in \mathfrak{R}$ and $U \in \mathfrak{U}$,*

(iii) *$g(C) = h(C)$ for $C \in \mathfrak{C}$.*

Moreover $\|g\| = \|h\|$, and if h is positive then g is positive.

(2) *There is a unique linear mapping $T: \mathfrak{R} \to \mathfrak{C}$ such that*

(i) *T is ultraweakly continuous,*

(ii) *T is positive, and $T(I) = I$,*

(iii) *$T(U^*AU) = T(A)$ for $A \in \mathfrak{R}$ and $U \in \mathfrak{U}$,*

(iv) *$T(CA) = C \cdot T(A)$ for $A \in \mathfrak{R}$ and $C \in \mathfrak{C}$.*

The terminology is that of [3], except that *finite* is used here in the sense that if E is any projection in $\mathfrak{R}$ that is equivalent to I then $E = I$. A positive linear form g on $\mathfrak{R}$ satisfying (i) and (ii) of (1) is called a *finite normal trace* on $\mathfrak{R}$. The mapping T in part (2) is the *canonical centre-valued trace* of $\mathfrak{R}$.

The "uniqueness" part of (1) and the deduction of (2) from (1) are straightforward. The "existence" part of (1) will be proved by the application of a fixed point theorem. We first require two lemmas.

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LEMMA 1. *Let E and F be projections in a finite von Neumann algebra $\mathfrak{R}$ and let (E_k) be a sequence of projections in $\mathfrak{R}$ such that* $\sup E_k = E$, *and $E_k \leqq E_{k+1}$, $E_k \prec F$ for all k. Then $E \prec F$.*

PROOF. Let $E_1 \sim F_1 \leqq F$. Let $k \geqq 1$ and suppose that projections $F_1 \cdots F_k \in \mathfrak{R}$ have been chosen so that $F_iF_j = 0$ for $1 \leqq i < j \leqq k$, $\sum_{i=1}^k F_i \leqq F$, and $E_{i+1} - E_i \sim F_{i+1}$ for $1 \leqq i \leqq k-1$. Since $E_{k+1} \prec F$ and $\mathfrak{R}$ is finite, $I - E_{k+1} \succ I - F$. Since also $E_k \sim \sum_{i=1}^k F_i$, it follows that

$$I - (E_{k+1} - E_k) \succ I - \left(F - \sum_{i=1}^k F_i\right), \qquad E_{k+1} - E_k \prec F - \sum_{i=1}^k F_i,$$

and we can choose F_{k+1} so that $E_{k+1} - E_k \sim F_{k+1} \leqq F - \sum_{i=1}^k F_i$. Thus there is a sequence of projections (F_k) such that $F_iF_j = 0$ for $i \neq j$, $\sum_{i=1}^\infty F_i \leqq F$, and $E_{k+1} - E_k \sim F_{k+1}$ for all k. Hence

$$E = E_1 + \sum_{i=1}^\infty (E_{i+1} - E_i) \sim \sum_{i=1}^\infty F_i \leqq F.$$

Let $\mathfrak{R}_*$ denote the dual of $\mathfrak{R}$ for the ultraweak topology, and for each $U \in \mathfrak{U}$, let T_U be the linear isometry of $\mathfrak{R}_*$ onto itself such that $(T_Uf)(A) = f(U^*AU)$ for all $f \in \mathfrak{R}_*$ and $A \in \mathfrak{R}$.

LEMMA 2. *Let $\mathfrak{R}$ be a finite von Neumann algebra, let $f \in \mathfrak{R}_*$, and let Q be the closed convex hull in $\mathfrak{R}_*$ of the set $K = \{T_Uf : U \in \mathfrak{U}\}$. Then Q is weakly compact.*

PROOF. By [4, V.6.4] it is sufficient to show that K is weakly relatively compact. If K is not weakly relatively compact, then by [1, Theorem II.2(2)] there is a sequence (E_n) of mutually orthogonal projections in $\mathfrak{R}$, a sequence (f_n) in K, and a real positive ϵ, such that $|f_n(E_n)| \geqq \epsilon$ for all n. Let $U_n \in \mathfrak{U}$ be such that $f_n = T_{U_n}f$, and let $F_n = U_n^*E_nU_n$, so that (F_n) is a sequence of projections in $\mathfrak{R}$ such that $F_n \sim E_n$ and $|f(F_n)| = |f(U_n^*E_nU_n)| = |(T_{U_n}f)(E_n)| = |f_n(E_n)| \geqq \epsilon$ for all n. Let $P_n = \sum_{m \geqq n} E_m$, $Q_n = \sup_{m \geqq n} F_m$, so that $P_{n+1} \leqq P_n$, $Q_{n+1} \leqq Q_n$ for all n, and let $G = \inf Q_n$. Let n now be fixed and for each k let $R_k = \sup\{F_i : n \leqq i \leqq n+k\}$. Suppose that $k \geqq 1$ and $R_{k-1} \prec \sum_{i=n}^{n+k-1} E_i$. Now

$$R_k = R_{k-1} + (\sup\{R_{k-1}, F_{n+k}\} - R_{k-1})$$

and

$$\sup\{R_{k-1}, F_{n+k}\} - R_{k-1} \sim F_{n+k} - \inf\{R_{k-1}, F_{n+k}\} \leqq F_{n+k} \sim E_{n+k},$$

by [3, III.1.1, Corollary 1]. Hence $R_k \prec \sum_{i=n}^{n+k} E_i$, and it follows that $R_k \prec \sum_{i=n}^{n+k} E_i \leqq P_n$ for all k, and by Lemma 1, that $Q_n = \sup R_k \prec P_n$.

Since $\mathfrak{R}$ is finite, $I-P_n \prec I-Q_n \leqq I-G$ for all n, and again by Lemma 1, $I=\sup(I-P_n)\prec I-G$, whence $G=0$. Hence (F_n) converges ultraweakly to 0, which contradicts $|f(F_n)| \geqq \epsilon > 0$ for large n.

There is an obvious analogy between the above method of proof and the statement of [**10**, Theorem 8].

Proof of Theorem. (1) Let $f \in \mathfrak{R}_*$ be chosen so that $f(C)=h(C)$ for $C \in \mathcal{C}$, let Q be the set defined in Lemma 2, and let $\mathcal{S}$ be the group $\{T_U: U \in \mathfrak{U}\}$ acting on Q. The set Q is weakly compact by Lemma 2, and $\mathcal{S}$ is obviously noncontracting in the sense of [**11**, Definition]. Hence, by the Ryll-Nardzewski fixed point theorem [**11**, Theorem 3], [**9**], where we take the locally convex space E to be $\mathfrak{R}_*$ with the norm topology, there is an element $g \in Q$ such that $T_U g = g$ for all $U \in \mathfrak{U}$, that is, $g(U^*AU)=g(A)$ for all $A \in \mathfrak{R}$ and $U \in \mathfrak{U}$. If $C \in \mathcal{C}$, then, for all $U \in \mathfrak{U}$, $U^*CU=C$, $(T_U f)(C)=f(U^*CU)=f(C)=h(C)$, $k(C)=h(C)$ for any $k \in Q$, hence $g(C)=h(C)$.

Now let g be any linear form on $\mathfrak{R}$ satisfying (i), (ii), (iii), and let $g=|g|\cdot V$ be the polar decomposition of g. Then for any $U \in \mathfrak{U}$, $T_U g=(T_U|g|)\cdot(UVU^*)$ is the polar decomposition of $T_U g$ $(=g)$. By uniqueness of the polar decomposition, $UVU^*=V$ for all $U \in \mathfrak{U}$, so that $V \in \mathcal{C}$ and $\|g\|=g(V^*)=h(V^*)\leqq\|h\|$. Since obviously $\|g\|\geqq\|h\|$, we have $\|g\|=\|h\|$. An application of the preceding argument with $h=0$ suffices to prove uniqueness. If h is positive, then $g(1)=h(1)=\|h\|=\|g\|$, and so g is positive.

(2) By part (1) we can define a linear isometry $T_*: \mathcal{C}_* \to \mathfrak{R}_*$ such that

(a) $(T_*h)(U^*AU)=(T_*h)(A)$,

(b) $(T_*h)(C)=h(C)$,

for $h \in \mathcal{C}_*$, $A \in \mathfrak{R}$, $U \in \mathfrak{U}$ and $C \in \mathcal{C}$. Let $T: \mathfrak{R} \to \mathcal{C}$ be the conjugate mapping. Since the ultraweak topology agrees with the weak* topology when $\mathfrak{R}$ is identified with the dual of $\mathfrak{R}_*$, (i) is immediate. (ii) and (iii) are easily verified; (iv) and the uniqueness of T follow from the uniqueness proved in (1).

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