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APPLICATIONS OF THE SEMISIMPLE SPLITTING

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Let S be a solvable simply connected analytic group. A closed subgroup C of S will be called uniform in S if the quotient space S/C is compact. We will make the assumption throughout the rest of the paper that a closed uniform subgroup C of S has the property that it contains no normal analytic subgroup of S. Let C be a closed uniform subgroup of S. In [4], G. D. Mostow proved that the intersection of C with the nil-radical of S must be uniform in the nil-radical of S. L. Auslander in [6], [7] exploited this fact to relate the structure of C to the structure of S. The object of this paper is to show how the semisimple splitting of S, introduced in [1] and studied further in [2] and [3], provides a convenient language with which to deal with the main theorems of [4], [6] and [7].

Preliminaries. We shall need the following ideas from the theory of algebraic groups. Let X, Y, and Z be subgroups of $\mathrm{Gl}(n, R)$, the group of all n by n real matrices. We denote the algebraic hull of X by $\mathfrak{A}(X)$, and the group of commutators of X and Y by (X, Y). Then

(a) $\mathfrak{A}((X, Y)) = \mathfrak{A}((\mathfrak{A}(X), \mathfrak{A}(Y)))$.

(b) If X is a solvable algebraic group then we can write

$$X = U \cdot T \quad \text{(semidirect product)}$$

where U is the collection of all unipotent matrices in X and T is a maximal completely reducible subgroup of X. Moreover if $T^\#$ is another maximal completely reducible subgroup of X then there is an x in U such that $xTx^{-1} = T^\#$.

We will require also the following facts from the theory of nilpotent Lie groups. By a real nilpotent group N we mean a simply connected nilpotent analytic group. We denote the Lie algebra of N by $L(N)$. The exponential map defines a homeomorphism of $L(N)$ onto N. By a lattice of a vector space V we mean the collection of all integer combinations of a basis of V. A discrete uniform subgroup C of N is called

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a lattice of N if it is the exponential of a lattice in $L(N)$. Given a discrete uniform subgroup C of N and a group A of automorphisms of N such that $A(C)=C$, there exists a lattice $C^{\#}$ of N containing C as a subgroup of finite index such that $A(C^{\#})=C^{\#}$. We will also assume that the reader is familiar with the work of Mal'cev [**5**] whose results we will often state in the language of algebraic groups.

The semisimple splitting. Let S be a simply connected solvable analytic group having nil-radical H. We will state those properties of the semisimple splitting of S needed in what follows.

THEOREM 1. *There exists a unique group $S^{\#}$ containing S as a normal subgroup with the following properties:*

(a) *$S^{\#}$ has a semidirect product decomposition $N\cdot T$ where N is the nil-radical of $S^{\#}$, and T is an abelian analytic group acting faithfully on N as a group of semisimple automorphisms. Any such decomposition of $S^{\#}$, $S^{\#}=N\cdot T$ will be called a Mal'cev decomposition of $S^{\#}$ and T will be called a Mal'cev factor of $S^{\#}$. We denote the projection of $S^{\#}$ onto N by P_T and the projection of $S^{\#}$ onto T by t_T.*

(b) *Let $S^{\#}=N\cdot T=N\cdot T'$ be two Mal'cev decompositions of $S^{\#}$. Then there is an x in H such that $xTx^{-1}=T'$.*

(c) *Let $S^{\#}=N\cdot T$ be a Mal'cev decomposition of $S^{\#}$. Then the restriction of P_T to S is a homeomorphism of S onto N.*

(d) *S and N generate $S^{\#}$.*

(e) *H is contained in N and $S^{\#}/H$ is abelian.*

(f) *There exists a faithful finite dimensional representation*

$$\beta: S^{\#} \to \mathrm{Gl}(n, R)$$

such that $\beta(N)$ is the collection of all unipotent automorphisms in $\mathfrak{A}(\beta(S^{\#}))$, and for any Mal'cev factor T of $S^{\#}$, $\beta(T)$ is a group of semisimple matrices.

Statements (a)–(e) can be found in [**1**]. Statement (f) follows from the Birkhoff imbedding theorem and can be found in [**9**].

Let C be a closed uniform subgroup of S. By (f) we lose no generality if we assume in all that follows that $S^{\#}$ is a matrix group. By (b) of the preliminary section we can write $\mathfrak{A}(C)=U_C\cdot T_C$ where U_C is the collection of all unipotent matrices in $\mathfrak{A}(C)$ and T_C is a maximal completely reducible subgroup. Let T' be a maximal completely reducible subgroup of $\mathfrak{A}(S)$ containing T_C. Since maximal completely reducible subgroups of $\mathfrak{A}(S)$ are conjugate over (S, S) we can find a Mal'cev factor T of $S^{\#}$ such that $T\subset T'$. Hence we can write a Mal'cev decomposition $S^\# = N\cdot T$ where $t_T(C)\subset \mathcal{A}(C)$. We shall say in this case that the Mal'cev decomposition $N\cdot T$ of $S^\#$ is compatible with C.

Mostow's theorem. We shall assume the following simple lemma. (See Lemma 6 in [4].)

LEMMA 1. *Let Δ be a lattice in a vector space V. Then there is an $\epsilon > 0$ depending solely on the dimension of V such that if A is an automorphism of V such that $A(\Delta) = \Delta$ and all the eigenvalues of A are within ϵ of one, then $A = I$, where I denotes the identity map on V.*

LEMMA 2. *Let S be a simply connected solvable analytic group with nil-radical H. Then if S has a discrete uniform subgroup C such that $S = (HC)^-$, S is nilpotent. We are using a bar to the superior of a subgroup of S to denote the closure of the subgroup in S.*

PROOF. Let $N\cdot T$ be a Mal'cev decomposition of $S^\#$ compatible with C. Let $M = \mathcal{A}(C\cap H)$ and $C' = MC$. Since C'/C is compact, C' is a closed subgroup of S. Hence $P_T(C')$ is a closed uniform subgroup of N that is invariant under $t_T(C')$. The connected component of the identity of $P_T(C')$ is M. Moreover M is a normal subgroup of $S^\#$. Since T is completely reducible we must show that $\mathrm{ad}_M T = I$ and that T induces the identity on N/M. If $\mathrm{ad}_M T \neq I$ then there is an element c in C such that $\mathrm{ad}_M c$ is not unipotent but has eigenvalues arbitrarily close to one. But $\mathrm{ad}_M c$ takes $C\cap H$ onto itself, hence we contradict Lemma 1. Thus $\mathrm{ad}_M T = I$. Since $P_T(C')/M$ is a discrete uniform subgroup of N/M, $(C, P_T(C'))\subset M$ implies that T induces the identity on N/M.

LEMMA 3. *Let S be a simply connected solvable analytic group with nil-radical H. Let C be a closed uniform subgroup of S. Then the identity component C_0 of C is contained in H. Moreover if C/C_0 is nilpotent, then C is nilpotent.*

PROOF. Let $N\cdot T$ be a Mal'cev decomposition of $S^\#$ compatible with C. Since C is uniform in S it follows that $N\subset \mathcal{A}(C)$. Hence C_0 is normalized by N and $C_0\cap H$ is a normal analytic subgroup of N. Take x in C_0 and write $x = nt$ where n is in N and t is in T. Let K be the ideal in N generated by $(T-I)N$. Then K is a normal analytic subgroup of $S^\#$ contained in $C_0\cap H$. Hence since we are assuming that C_0 contains no nontrivial normal analytic subgroup of S we have that $t = I$. Then $\mathrm{ad}_H C$ consists of unipotent automorphisms and HC_0 is a normal nilpotent analytic subgroup of S. Hence $C_0\subset H$. Assume now that C/C_0 is nilpotent. Then $\mathcal{A}(C)/C_0$ is nilpotent. Take x in C and write $x = nt$.

Then $(t-I)N \subset C_0 \cap H$ and we can argue as before to conclude that C operates on N by unipotent automorphisms. Hence C is nilpotent.

THEOREM 2. *Let C be a closed uniform subgroup of a simply connected solvable analytic group S. Let H be the nil-radical of S. Then HC is a closed subgroup of S. Moreover $C_0 \subset H$.*

PROOF. Let G be the identity component of $(HC)^-$. We must show that $H=G$. Let $C'=C\cap G$. Then C' is a closed uniform subgroup of S and $S=(HC')^-$. Clearly C_0 is a normal subgroup of G. Hence by Lemma 2, G/C_0 is nilpotent. Arguing as in Lemma 3, we can show that $\mathrm{ad}_H C'$ consists of unipotent automorphisms. Hence G is nilpotent.

Auslander's theorems. Let Z^n denote the cartesian product of the integers taken n times. Let R^n denote the cartesian product of the reals taken n times.

THEOREM 3. *A necessary and sufficient condition for a simply connected solvable analytic group S to contain Z^n as a discrete uniform subgroup is that S satisfy the diagram*

$$1 \rightarrow R^s \rightarrow S \rightarrow R^t \rightarrow 1$$

where

(a) $s+t=n$,

(b) *the extension is the split extension,*

(c) *the automorphisms of R^s induced by R^t form a compact group.*

PROOF. Assume that S contains Z^n as a discrete uniform subgroup. Let $N\cdot T$ be a Mal'cev decomposition of $S^\#$ compatible with Z^n. Since $P_T(Z^n)$ is a discrete uniform subgroup of N centralized by $\mathfrak{A}(Z^n)$, we have that Z^n is a discrete uniform subgroup of N. Hence N is abelian and T is compact. From $(N, T)\subset H$ and T being completely reducible, it follows that there is a subspace $N^\#$ of N such that $N=N^\# \oplus H$ as a vector space and $(T, N^\#)=(1)$. Let $S'=P_T^{-1}(N^\#)$. Then $S=H\cdot S'$ as a semidirect product.

The converse is trivial.

THEOREM 4. *Let S be a simply connected solvable analytic group and let C be a closed uniform subgroup of S. Assume that C/C_0 is nilpotent. Then there is a real nilpotent group N such that C is contained in N as a closed uniform subgroup, and a compact abelian analytic group T of automorphisms of N such that $S\subset N\cdot T$. In fact $S^\#=N\cdot T$.*

PROOF. Since C/C_0 is nilpotent, Lemma 3 implies that C is nilpotent. Moreover if $N \cdot T$ is a Mal'cev decomposition of $S^{\#}$ compatible with C we have also that $C \subset N$ as a closed uniform subgroup.

In terms of the semisimple splitting many cohomological arguments in solvable Lie theory depend upon the following lemma. (See Lemma 2.1 of [2].)

LEMMA 4. *Let T be an abelian analytic group of semisimple automorphisms of a vector space V. Assume that the Fitting one-space of T in V is trivial. Then the first cohomology group vanishes, i.e. $H^1(T, V) = 0$.*

The following theorem is easily seen to imply Theorem 2 of [7].

THEOREM 5. *Let N be a real nilpotent group and let T_i, $i=1, 2$, be abelian analytic groups of semisimple automorphisms of N. Let H be a normal analytic subgroup of N invariant under T_i, $i=1, 2$, such that,*

(a) *N/H is abelian,*

(b) *T_i, $i=1, 2$, induce the identity map on N/H,*

(c) *T_1 restricted to H is equal to T_2 restricted to H. Let $\alpha: T_1 \rightarrow T_2$ be the induced isomorphism.*

(d) *Let M be the last term in the lower central series of N. Then the group of automorphisms on N/M induced by T_1 and T_2 coincide.*

Then there is an isomorphism $\bar{\alpha}$ of N such that the map

$$\bar{\bar{\alpha}}: N \cdot T_1 \rightarrow N \cdot T_2$$

given by $\bar{\bar{\alpha}}(n, t) = (\bar{\alpha}(n), \alpha(t))$ for n in N and t in T_1, defines an isomorphism of the semidirect product $N \cdot T_1$ onto the semidirect product $N \cdot T_2$.

PROOF. Choose x in N such that $(T, x) = (1)$. Consider the map $\eta: T_2 \rightarrow M$ given by $t(x) = \eta(t) \cdot x$ for t in T_2. Write $M = M_1 \oplus M_2$ where M_1 is the Fitting one-space of T and M_2 is a subspace of M complementary to M_1 and invariant under T_2. Consider the maps $\eta_i: T_2 \rightarrow M_i$, given by $\eta(t) = \eta_1(t)\eta_2(t)$. Then η_2 is a cocycle from T_2 to M_2. Thus there is an h in M_2 such that $\eta_2(t) = (t, h)$ for all t in T_2. Let $x' = h^{-1}x$. Then it is easy to see that $(T_2, x') = (1)$. Now let X be a subspace of N such that $N = H \oplus X$, and $(T, X) = (1)$. Let x_i, $i = 1, \cdots, n$, be a basis for X. Choose h_i in M such that $(T_2, h_i^{-1}x_i) = (1)$. Clearly there is an isomorphism $\bar{\alpha}$ of N which maps x_i onto $h_i^{-1}x_i$ and which acts by the identity map on H.

REFERENCES

1. L. Auslander and L. Green, *G-induced flows*, Amer. J. Math. **88** (1966), 43–60. MR **33** #7456.

2. L. Auslander and J. Brezin, *Almost algebraic Lie algebras*, J. Algebra **8** (1968), 295–313. MR **37** #344.

3. R. Tolimieri, *Foundations of solvable Lie groups*, J. Algebra (to appear).

4. G. D. Mostow, *Factor spaces of solvable groups*, Ann. of Math. (2) **60** (1954), 1–27. MR **15**, 853.

5. A. Mal'cev, *On a class of homogeneous spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 9–32; English transl., Amer. Math. Soc. Transl. (1) **9** (1962), 276–307. MR **10**, 507.

6. L. Auslander and M. Auslander, *Solvable Lie groups and locally euclidean Riemann spaces*, Proc. Amer. Math. Soc. **8** (1958), 933–941. MR **21** #2021.

7. L. Auslander, *Solvable Lie groups acting on nilmanifolds*, Amer. J. Math. **82** (1960), 653–660. MR **23** #A241.

8. ———, *On a problem of Phillip Hall*, Ann. of Math. (2) **86** (1967), 112–116. MR **36** #1540.

9. G. Birkhoff, *Representability of Lie algebras and Lie groups by matrices*, Ann. of Math. **38** (1937).

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