 APPLICATIONS OF THE SEMISIMPLE SPLITTING

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Communicated by L. Auslander, September 2, 1970

Let $S$ be a solvable simply connected analytic group. A closed subgroup $C$ of $S$ will be called uniform in $S$ if the quotient space $S/C$ is compact. We will make the assumption throughout the rest of the paper that a closed uniform subgroup $C$ of $S$ has the property that it contains no normal analytic subgroup of $S$. Let $C$ be a closed uniform subgroup of $S$. In [4], G. D. Mostow proved that the intersection of $C$ with the nil-radical of $S$ must be uniform in the nil-radical of $S$. L. Auslander in [6], [7] exploited this fact to relate the structure of $C$ to the structure of $S$. The object of this paper is to show how the semisimple splitting of $S$, introduced in [1] and studied further in [2] and [3], provides a convenient language with which to deal with the main theorems of [4], [6] and [7].

Preliminaries. We shall need the following ideas from the theory of algebraic groups. Let $X$, $Y$, and $Z$ be subgroups of $GL(n, R)$, the group of all $n$ by $n$ real matrices. We denote the algebraic hull of $X$ by $\mathfrak{a}(X)$, and the group of commutators of $X$ and $Y$ by $(X, Y)$. Then

(a) $\mathfrak{a}((X, Y)) = \mathfrak{a}((\mathfrak{a}(X), \mathfrak{a}(Y)))$.

(b) If $X$ is a solvable algebraic group then we can write $X = U \cdot T$ (semidirect product)

where $U$ is the collection of all unipotent matrices in $X$ and $T$ is a maximal completely reducible subgroup of $X$. Moreover if $T'$ is another maximal completely reducible subgroup of $X$ then there is an $x$ in $U$ such that $xTx^{-1} = T'$.

We will require also the following facts from the theory of nilpotent Lie groups. By a real nilpotent group $N$ we mean a simply connected nilpotent analytic group. We denote the Lie algebra of $N$ by $L(N)$. The exponential map defines a homeomorphism of $L(N)$ onto $N$. By a lattice of a vector space $V$ we mean the collection of all integer combinations of a basis of $V$. A discrete uniform subgroup $C$ of $N$ is called

AMS 1969 subject classifications. Primary 2050; Secondary 2048, 1450.

Key words and phrases. Algebraic group, the algebraic hull, unipotent matrix, semisimple matrix completely reducible group, nilpotent group, nil-radical solvable group, simply connected, lattice, discrete, uniform, semisimple splitting, Birkhoff embedding theorem, eigenvalues semidirect product, Fitting one-space, first cohomology group.

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a lattice of $N$ if it is the exponential of a lattice in $L(N)$. Given a discrete uniform subgroup $C$ of $N$ and a group $A$ of automorphisms of $N$ such that $A(C) = C$, there exists a lattice $C'$ of $N$ containing $C$ as a subgroup of finite index such that $A(C') = C'$. We will also assume that the reader is familiar with the work of Mal'cev [5] whose results we will often state in the language of algebraic groups.

The semisimple splitting. Let $S$ be a simply connected solvable analytic group having nil-radical $H$. We will state those properties of the semisimple splitting of $S$ needed in what follows.

**Theorem 1.** There exists a unique group $S^\#$ containing $S$ as a normal subgroup with the following properties:

(a) $S^\#$ has a semidirect product decomposition $N \cdot T$ where $N$ is the nil-radical of $S^\#$, and $T$ is an abelian analytic group acting faithfully on $N$ as a group of semisimple automorphisms. Any such decomposition of $S^\#$, $S^\# = N \cdot T$ will be called a Mal'cev decomposition of $S^\#$ and $T$ will be called a Mal'cev factor of $S^\#$. We denote the projection of $S^\#$ onto $N$ by $P_T$ and the projection of $S^\#$ onto $T$ by $t_T$.

(b) Let $S^\# = N \cdot T = N \cdot T'$ be two Mal'cev decompositions of $S^\#$. Then there is an $x$ in $H$ such that $xT^x^{-1} = T'$.

(c) Let $S^\# = N \cdot T$ be a Mal'cev decomposition of $S^\#$. Then the restriction of $P_T$ to $S$ is a homeomorphism of $S$ onto $N$.

(d) $S$ and $N$ generate $S^\#$.

(e) $H$ is contained in $N$ and $S^\#/H$ is abelian.

(f) There exists a faithful finite dimensional representation

$$\beta : S^\# \to \text{Gl}(n, R)$$

such that $\beta(N)$ is the collection of all unipotent automorphisms in $\alpha(\beta(S^\#))$, and for any Mal'cev factor $T$ of $S^\#$, $\beta(T)$ is a group of semisimple matrices.

Statements (a)-(e) can be found in [1]. Statement (f) follows from the Birkhoff imbedding theorem and can be found in [9].

Let $C$ be a closed uniform subgroup of $S$. By (f) we lose no generality if we assume in all that follows that $S^\#$ is a matrix group. By (b) of the preliminary section we can write $\alpha(C) = U_C \cdot T_C$ where $U_C$ is the collection of all unipotent matrices in $\alpha(C)$ and $T_C$ is a maximal completely reducible subgroup. Let $T'$ be a maximal completely reducible subgroup of $\alpha(S)$ containing $T_C$. Since maximal completely reducible subgroups of $\alpha(S)$ are conjugate over $(S, S)$ we can find a Mal'cev factor $T$ of $S^\#$ such that $T \subset T'$. Hence we can write a Mal'cev de-
composition $S\dagger = N \cdot T$ where $t_T(C) \subset \alpha(C)$. We shall say in this case that the Mal’cev decomposition $N \cdot T$ of $S\dagger$ is compatible with $C$.

**Mostow’s theorem.** We shall assume the following simple lemma. (See Lemma 6 in [4].)

**Lemma 1.** Let $\Delta$ be a lattice in a vector space $V$. Then there is an $\epsilon > 0$ depending solely on the dimension of $V$ such that if $A$ is an automorphism of $V$ such that $A(\Delta) = \Delta$ and all the eigenvalues of $A$ are within $\epsilon$ of one, then $A = I$, where $I$ denotes the identity map on $V$.

**Lemma 2.** Let $S$ be a simply connected solvable analytic group with nil-radical $H$. Then if $S$ has a discrete uniform subgroup $C$ such that $S = (HC)^\dagger$, $S$ is nilpotent. We are using a bar to the superior of a subgroup of $S$ to denote the closure of the subgroup in $S$.

**Proof.** Let $N \cdot T$ be a Mal’cev decomposition of $S\dagger$ compatible with $C$. Let $M = \alpha(C \cap H)$ and $C' = MC$. Since $C'/C$ is compact, $C'$ is a closed subgroup of $S$. Hence $P_T(C')$ is a closed uniform subgroup of $N$ that is invariant under $t_T(C')$. The connected component of the identity of $P_T(C')$ is $M$. Moreover $M$ is a normal subgroup of $S\dagger$. Since $T$ is completely reducible we must show that $\text{ad}_M T = I$ and that $T$ induces the identity on $N/M$. If $\text{ad}_M T \neq I$ then there is an element $c$ in $C$ such that $\text{ad}_MC$ is not unipotent but has eigenvalues arbitrarily close to one. But $\text{ad}_MC$ takes $C \cap H$ onto itself, hence we contradict Lemma 1. Thus $\text{ad}_M T = I$. Since $P_T(C')/M$ is a discrete uniform subgroup of $N/M$, $(C, P_T(C')) \subset M$ implies that $T$ induces the identity on $N/M$.

**Lemma 3.** Let $S$ be a simply connected solvable analytic group with nil-radical $H$. Let $C$ be a closed uniform subgroup of $S$. Then the identity component $C_0$ of $C$ is contained in $H$. Moreover if $C/C_0$ is nilpotent, then $C$ is nilpotent.

**Proof.** Let $N \cdot T$ be a Mal’cev decomposition of $S\dagger$ compatible with $C$. Since $C$ is uniform in $S$ it follows that $N \subset \alpha(C)$. Hence $C_0$ is normalized by $N$ and $C_0 \cap H$ is a normal analytic subgroup of $N$. Take $x$ in $C_0$ and write $x = nt$ where $n$ is in $N$ and $t$ is in $T$. Let $K$ be the ideal in $N$ generated by $(T-I)N$. Then $K$ is a normal analytic subgroup of $S\dagger$ contained in $C_0 \cap H$. Hence since we are assuming that $C_0$ contains no nontrivial normal analytic subgroup of $S$ we have that $t = I$. Then $\text{ad}_NC$ consists of unipotent automorphisms and $HC_0$ is a normal nilpotent analytic subgroup of $S$. Hence $C_0 \subset H$. Assume now that $C/C_0$ is nilpotent. Then $\alpha(C)/C_0$ is nilpotent. Take $x$ in $C$ and write $x = nt$. 

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Then \((t-I)N \subset C_0 \cap H\) and we can argue as before to conclude that \(C\) operates on \(N\) by unipotent automorphisms. Hence \(C\) is nilpotent.

**Theorem 2.** Let \(C\) be a closed uniform subgroup of a simply connected solvable analytic group \(S\). Let \(H\) be the nil-radical of \(S\). Then \(HC\) is a closed subgroup of \(S\). Moreover \(C_0 \subset H\).

**Proof.** Let \(G\) be the identity component of \((HC)^-\). We must show that \(H = G\). Let \(C' = C \cap G\). Then \(C'\) is a closed uniform subgroup of \(S\) and \(S = (HC')^-\). Clearly \(C_0\) is a normal subgroup of \(G\). Hence by Lemma 2, \(G/C_0\) is nilpotent. Arguing as in Lemma 3, we can show that \(\text{ad}_H C'\) consists of unipotent automorphisms. Hence \(G\) is nilpotent.

**Auslander's theorems.** Let \(Z^n\) denote the cartesian product of the integers taken \(n\) times. Let \(R^n\) denote the cartesian product of the reals taken \(n\) times.

**Theorem 3.** A necessary and sufficient condition for a simply connected solvable analytic group \(S\) to contain \(Z^n\) as a discrete uniform subgroup is that \(S\) satisfy the diagram

\[
1 \rightarrow R^s \rightarrow S \rightarrow R^t \rightarrow 1
\]

where

(a) \(s + t = n\),
(b) the extension is the split extension,
(c) the automorphisms of \(R^s\) induced by \(R^t\) form a compact group.

**Proof.** Assume that \(S\) contains \(Z^n\) as a discrete uniform subgroup. Let \(N \cdot T\) be a Mal'cev decomposition of \(S^\#\) compatible with \(Z^n\). Since \(P_T(Z^n)\) is a discrete uniform subgroup of \(N\) centralized by \(\alpha(Z^n)\), we have that \(Z^n\) is a discrete uniform subgroup of \(N\). Hence \(N\) is abelian and \(T\) is compact. From \((N, T) \subset H\) and \(T\) being completely reducible, it follows that there is a subspace \(N^\#\) of \(N\) such that \(N = N^\# \oplus H\) as a vector space and \((T, N^\#) = (1)\). Let \(S' = P_T^{-1}(N^\#)\). Then \(S = H \cdot S'\) as a semidirect product.

The converse is trivial.

**Theorem 4.** Let \(S\) be a simply connected solvable analytic group and let \(C\) be a closed uniform subgroup of \(S\). Assume that \(C/C_0\) is nilpotent. Then there is a real nilpotent group \(N\) such that \(C\) is contained in \(N\) as a closed uniform subgroup, and a compact abelian analytic group \(T\) of automorphisms of \(N\) such that \(S \subset N \cdot T\). In fact \(S^\# = N \cdot T\).

**Proof.** Since \(C/C_0\) is nilpotent, Lemma 3 implies that \(C\) is nil-
potent. Moreover if \(N \cdot T\) is a Mal'cev decomposition of \(S\) compatible with \(C\) we have also that \(C \subset N\) as a closed uniform subgroup.

In terms of the semisimple splitting many cohomological arguments in solvable Lie theory depend upon the following lemma. (See Lemma 2.1 of [2].)

**Lemma 4.** Let \(T\) be an abelian analytic group of semisimple automorphisms of a vector space \(V\). Assume that the Fitting one-space of \(T\) in \(V\) is trivial. Then the first cohomology group vanishes, i.e. \(H^1(T, V) = 0\).

The following theorem is easily seen to imply Theorem 2 of [7].

**Theorem 5.** Let \(N\) be a real nilpotent group and let \(T_i, i = 1, 2,\) be abelian analytic groups of semisimple automorphisms of \(N\). Let \(H\) be a normal analytic subgroup of \(N\) invariant under \(T_i, i = 1, 2,\) such that,

(a) \(N/H\) is abelian,
(b) \(T_i, i = 1, 2,\) induce the identity map on \(N/H,\)
(c) \(T_1\) restricted to \(H\) is equal to \(T_2\) restricted to \(H\). Let \(\alpha: T_1 \to T_2\) be the induced isomorphism.
(d) Let \(M\) be the last term in the lower central series of \(N\). Then the group of automorphisms on \(N/M\) induced by \(T_1\) and \(T_2\) coincide.

Then there is an isomorphism \(\bar{\alpha}\) of \(N\) such that the map

\[ \bar{\alpha}: N \cdot T_1 \to N \cdot T_2 \]

given by \(\bar{\alpha}(n, t) = (\alpha(n), \alpha(t))\) for \(n\) in \(N\) and \(t\) in \(T_1\), defines an isomorphism of the semidirect product \(N \cdot T_1\) onto the semidirect product \(N \cdot T_2\).

**Proof.** Choose \(x\) in \(N\) such that \((T, x) = (1)\). Consider the map \(\eta: T_2 \to M\) given by \(t(x) = \eta(t) \cdot x\) for \(t\) in \(T_2\). Write \(M = M_1 \oplus M_2\) where \(M_1\) is the Fitting one-space of \(T\) and \(M_2\) is a subspace of \(M\) complementary to \(M_1\) and invariant under \(T_2\). Consider the maps \(\eta_i: T_2 \to M_i\), given by \(\eta(t) = \eta_1(t) \eta_2(t)\). Then \(\eta_2\) is a cocycle from \(T_2\) to \(M_2\). Thus there is an \(h\) in \(M_2\) such that \(\eta_2(t) = (t, h)\) for all \(t\) in \(T_2\). Let \(x' = h^{-1}x\). Then it is easy to see that \((T_2, x') = (1)\). Now let \(X\) be a subspace of \(N\) such that \(N = H \oplus X\), and \((T, X) = (1)\). Let \(x_i, i = 1, \ldots, n,\) be a basis for \(X\). Choose \(h_i\) in \(M\) such that \((T_2, h_i^{-1}x_i) = (1)\). Clearly there is an isomorphism \(\bar{\alpha}\) of \(N\) which maps \(x_i\) onto \(h_i^{-1}x_i\) and which acts by the identity map on \(H\).

**References**

MR 33 #7456.


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