GENERATION OF EQUICONTINUOUS SEMIGROUPS
BY HERMITIAN AND SECTORIAL OPERATORS. II

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1. Introduction. This announcement concerns the topological aspects of the generation theory of equicontinuous semigroups and groups of operators on a complete complex locally convex space (lcs) $X$, and uses the recalibration theorem from [6] to relate these to the more geometrical aspects treated in [8] (with which the reader is assumed to be familiar). Perturbation techniques from [7], along with other devices, are used to develop applications to the theory of abstract heat equations and to the theory of distribution semigroups. Details will appear in [9].

2. Quasi-equicontinuous semigroups. The semigroups considered here generalize the contraction holomorphic semigroups $CH(\Phi, \Gamma)$ on a complete complex lcs $X$ discussed in [8], where for, $0 \leq \Phi \leq \pi/2$, $S_\Phi = \{ z \in \mathbb{C} : \arg z \leq \Phi \}$ and $\Delta_\Phi = \{ z \in \mathbb{C} : \pi/2 + \Phi \leq \arg z \leq 3\pi/2 - \Phi \}$.

**Definition 1.** Let $\omega \geq 0$. Then a family $\{ e^{-at}T_z : z \in S_\Phi \} \subset \mathcal{L}(X)$ of continuous linear transformations is a quasi-equicontinuous holomorphic semigroup of type $\omega$, or is in $EH(\Phi; \omega)$ if:

(a) it satisfies the usual algebraic, continuity and holomorphy conditions as in Definition (1a) of [8], and

(b) the family $\{ e^{-at}T_z : z \in S_\Phi \}$ is equicontinuous in $\mathcal{L}(X)$.

**Examples.** (1) If $\{ T_t : t \in [0, \infty) \}$ is a classical $C_0$ semigroup on a $B$-space [3], and $\omega > \omega_0 = \lim \{ (t^{-1} \log \| T_t \|) : t \to \infty \}$, then $\| T_t \| \leq Me^{\omega t}$ for suitable $M$ and the semigroup is in $EH(0; \omega)$ since operator-norm-bounded sets are equicontinuous. Similarly, every semigroup in Hille's class $H(-\Psi, \Psi)$ on a $B$-space [3] is in $EH(\Phi; \omega(\Phi))$ for every $\Phi < \Psi$ and suitable $\omega(\Phi)$.

(2) Every $CH(\Phi, \Gamma)$ semigroup from [8] is in $EH(\Phi; 0)$.

(3) Every equicontinuous $C_0$ semigroup as in Yosida [10] is in $EH(\Phi; \omega(\Phi))$ for every $\Phi < \Psi$ and suitable $\omega(\Phi)$.

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EH(0; 0); Miyadera [5] discusses EH(0; ω) semigroups on Fréchet spaces.

As in Definition 2 of [8], the (infinitesimal) generator A of an EH(Φ; ω) semigroup is defined, wherever the limit exists, by

\[ Au = \lim_{t \to 0} (T_t u - u) : t \to 0 \quad \text{in} \ [0, \infty] \subset S_\Phi \]

and the semigroup is smooth iff this limit exists for all \( u \in \mathfrak{X}. \)

**Theorem 1.** Let \( 0 \leq \Phi \leq \pi/2 \) and let \( \omega \geq 0. \) The following conditions on an operator A are equivalent.

(a) The domain \( D(A) \) is dense in \( \mathfrak{X}, \) A is closed, the finite spectrum of A, \( \sigma_F(A) \subset \Delta_\Phi + \omega \) (cf. [6]) and if \( d_\lambda = \mathrm{dist}(\lambda, \Delta_\Phi + \omega) \) then

\[ \{[d_\lambda(\lambda - A)^{-1}]^n : d_\lambda > 0, \ n = 0, 1, \ldots \} \]

is equicontinuous.

(b) The operator \( A \) is closed and densely defined, and \( A - \omega \) is \( \Phi \)-sectorial with respect to some calibration \( \Gamma \) for \( \mathfrak{X}. \)

(c) There exists a calibration \( \Gamma \) for \( \mathfrak{X} \) such that \( A - \omega \) generates a semigroup \( \{T_z : z \in S_\Phi \} \) in \( CH(\Phi, \Gamma). \)

(d) The operator \( A \) generates an EH(Φ; ω) semigroup \( \{T_t : t \in S_\Phi \}. \)

**Proof scheme.** All implications in (a) \( \iff \) (b) \( \iff \) (c) \( \iff \) (d) are routine from [6], [7], and [8], with the exception of (b) \( \Rightarrow \) (a), which requires recalibration by an equicontinuous simply closed convex balanced semigroup \( S \) of operators generated by the \( d_\lambda(\lambda - A)^{-1} \) for \( \lambda \in S_\Phi. \)

Similarly, a group \( \{T_t : t \in R \} \) is in \( EC_0(\omega) \) iff it is \( C_0 \) as in Definition 4 of [8] and \( \{e^{-\omega t}T_t : t \in R \} \) is equicontinuous. The analog of Theorem 1 is straightforward, with \( \sigma_F(A) \) and \( W(A, \Delta_\Phi) \) contained in the strip \( \{z : \mathrm{Re}(z) \leq \omega \} \) in (a) and (b) respectively. The case \( \omega = 0 \) is of primary interest.

**Theorem 2.** If \( A \) is as in Theorem 1, the following are equivalent.

(a) The finite spectrum of closed, densely defined \( A \) is pure imaginary, and \( \{[\mathrm{Re}(\lambda) |(\lambda - A)^{-1}]^n : n = 0, 1, \ldots, \mathrm{Re}(\lambda) \neq 0 \} \) is equicontinuous.

(b) The operator \( iA \) is closed, densely defined, and hermitian with respect to a suitable calibration \( \Gamma. \)

(c) \( A \) generates a generalized unitary group \( \{T_t : t \in R \} \) in \( RC_0(\Gamma) \) for some suitable \( \Gamma \).

(d) \( A \) generates an \( EC_0(0) \) group \( \{T_t : t \in R \}. \)

3. Semigroups on Banach sub-and-super-spaces. We show here that the properties of EH(Φ; ω) semigroups are entirely determined by their natural classical actions on certain appropriate classes of
Banach subspaces and super-spaces of the given lcs \( \mathfrak{X} \). To simplify the discussion, suppose that \( \mathfrak{X} \) admits a continuous norm, hence a calibration \( \Gamma \) consisting of norms. (Recalibration preserves this property.)

Then the super-spaces are formed by completing \( \mathfrak{X} \) with respect to a particular norm \( \rho \in \Gamma \) to obtain a \( B \)-space \( \mathfrak{X}_\rho \).

**Theorem 3.** Let \( \{ T_\zeta : \zeta \in S_\Phi \} \) be an operator-valued function. Then it is an \( \text{EH}(\Phi; \omega) \) semigroup iff there exists a norm-calibration \( \Gamma \) for \( \mathfrak{X} \) such that every \( T_\zeta \) has an extension-by-continuity \( T_\zeta^\rho \in \mathfrak{B}(\mathfrak{X}_\rho) \) and \( \{ e^{-u T_\zeta^\rho} : \zeta \in S_\Phi \} \) forms a \( \text{CH}(\Phi, \{ \rho \}) \) semigroup on \( \mathfrak{X}_\rho \) for all \( \rho \in \Gamma \).

The subspaces are defined in terms of functions \( F: \Gamma \to (0, \infty) \), letting \( \| u \|_F = \{ \sup F(\rho)^{-1} \rho(u) : \rho \in \Gamma \} \in [0, \infty) \) and

\[
\mathfrak{X}_F = \{ u \in \mathfrak{X} : \| u \|_F < 1 \}
\]

with bounded unit ball \( \mathfrak{B}_F = \{ u \in \mathfrak{X} : \| u \|_F \leq 1 \} \). Then

\[
\mathfrak{X} = \bigcup \{ \mathfrak{X}_F : F: \Gamma \to (0, \infty) \} = \bigcup \{ \mathfrak{B}_F : F: \Gamma \to (0, \infty) \}.
\]

**Theorem 4.** Let \( \{ T_\zeta : \zeta \in S_\Phi \} \) be an operator-valued function. Then it is an \( \text{EH}(\Phi; \omega) \) semigroup iff there exists a norm-calibration \( \Gamma \) for \( \mathfrak{X} \) satisfying the two conditions:

1. every \( T_\zeta \) leaves every \( \mathfrak{X}_F \) invariant and restricts to a \( T_\zeta^\rho \in \mathfrak{B}(\mathfrak{X}_\rho) \) with \( \| e^{-u T_\zeta^\rho} \|_F \leq 1 \), and
2. for some \( \| \cdot \|_F \)-closed, \( \Gamma \)-dense subspace \( \mathfrak{Y}_F \subset \mathfrak{X}_F \) invariant under the \( T_\zeta \), the restricted \( \{ e^{-u T_\zeta^\rho} : \zeta \in S_\Phi \} \subset \mathfrak{B}(\mathfrak{Y}_F) \) is a \( \text{CH}(\Phi, \{ \| \cdot \|_F \}) \) semigroup.

**Remark.** It is not yet clear whether \( \Gamma \) can be chosen so that \( \{ e^{-u T_\zeta^\rho} : \zeta \in S_\Phi \} \) is in \( \text{CH}(\Phi, \{ \| \cdot \|_F \}) \) on all of every \( \mathfrak{X}_F \), although it is necessarily operator-norm holomorphic there on \( \text{int}(S_\Phi) \).

The inevitable variants of these results are also true for groups.

4. **Exponential distribution semigroups.** Lions [4] has defined, and proved a “Hille-Yosida theorem” for, a generalized class of exponential distribution semigroups of type \( \leq \omega \), (or \( \text{ED}(0; \omega) \) semigroups) on a \( B \)-space \( \mathfrak{X} \). These are \( \mathfrak{B}(\mathfrak{X}) \)-valued distributions (where \( \mathfrak{B}(\mathfrak{X}) \) carries the norm topology) which restrict to algebra homomorphisms into \( \mathfrak{B}(\mathfrak{X}) \) from the convolution algebra \( \mathfrak{D}_0 \) of \( C^\infty \) functions \( \varphi \) compactly supported in \( [0, \infty) \), forming \( \{ T(\varphi) : \varphi \in \mathfrak{D}_0 \} \). In addition to technical hypotheses governing behavior near 0, one assumes that the map \( \varphi \mapsto "T(e^{-u \varphi}(0))" \) extends to a tempered distribution. These arise classically if \( \{ T_t : t \in [0, \infty) \} \) is a semigroup homomorphism which is strongly continuous on the open set \( (0, \infty) \) and satisfies the integra-
bility condition at 0: $\int_0^\infty \|Tu\| \, dt < \infty$ for all $u \in X$, as in the Hille-Phillips semigroups of class $(0, A)$ [3]. Then $\{ T(\varphi) : \varphi \in D_0 \}$ is defined by integration for $\varphi \in D_0$ and $u \in X$ by

$$T(\varphi)u = \int_0^\infty \varphi(t) Tu \, dt.$$  

A classical theorem of Feller [1], relating these more general classical semigroups to those in $\text{EH}(0; \omega)$, can be generalized using Theorems 3 and 4 along with results due to Fujiwara [2].

**Theorem 5.** If $\{ T(\varphi) : \varphi \in D_0 \}$ is in $\text{ED}(0; \omega)$, then there exists a dense subspace $Y \subset X$, which is a B-space with respect to a stronger norm $\| \cdot \|_\varphi \geq \| \cdot \|_i$, and a classical $C_0$ semigroup $\{ T_i : t \in [0, \infty) \}$ on $(Y, \| \cdot \|_\varphi)$ such that for all $u \in Y$ and $\varphi \in D_0$, (2) holds and characterizes $\{ T(\varphi) : \varphi \in D_0 \}$ uniquely.

**Remarks.** (1) Either $Y$ is first-category in $X$, or $Y = X$ as topological vector spaces, and in the latter case the $\text{ED}(0; \omega)$ semigroup is exactly the integrated form (2) of $\{ T_i : t \in [0, \infty) \}$.

(2) “Holomorphic” exponential distribution semigroups have been defined on sectors $S_\phi$ for $0 < \phi \leq \pi/2$ in the work of Fujiwara and others [2], leading naturally to a class of $\text{ED}(\phi; \omega)$ semigroups to which Theorem 5 easily extends.

**Generalization.** If $\mathcal{B}(X)$ in the definition of an $\text{ED}(\phi; \omega)$ semigroup is replaced by $\mathcal{B}(X)$ on a general locally convex space, the entire Lions theory admits a reasonable generalization to this setting. A satisfactory $\Gamma$-independent theory has not yet been obtained.

5. **Yosida's analytic continuation theorem.** The recalibration theorem can be used to clarify, strengthen, and trivially generalize an important theorem of Yosida (cf. §IX.10 of [10]) concerning continuation of an $\text{EH}(0; 0)$ semigroup from $[0, \infty)$ to a suitable $S_\phi$ for $\phi > 0$ in order to obtain an $\text{EH}(\phi; \omega)$ semigroup.

**Theorem 6.** Let $\{ T_i : t \in [0, \infty) \}$ be an $\text{EH}(0; \omega)$ semigroup with generator $A$. Then (a), (b) and (c) are equivalent, and imply (d).

(a) For all $t > 0$, $T_i X \subset D(A)$, and there exists a $c > 0$ such that $\{ [ct(A - \omega)e^{-\omega t}T_i]n : 0 < t \leq 1$ and $n = 0, 1, \ldots \}$ is an equicontinuous family of operators on $X$.

(b) For all $t > 0$, $T_i X \subset D(A)$ and there exists a $c > 0$ and a calibration $\Gamma$ for $X$ such that $AT_i \in \mathcal{F}(X)$ and $\| (A - \omega) T_i \|_\Gamma \leq (ct e^{-\omega})^{-1}$ for all $0 < t \leq 1$.

(c) For some calibration $\Gamma$ and $c > 0$ the series in the formula
(3) \[ T_z = e^{zt} \sum \left\{ \frac{(z - t)^n}{n!} [(A - \omega)e^{-\omega t}T_z]^n : n = 0, 1, \ldots \right\} \]

converges in the \[ \| \cdot \|_r \] sense whenever \[ |z - t| < \text{cte}^{-1} \].

(d) For some constant \( c > 0 \) and any \( \Phi \leq \sin^{-1}(ce^{-1}) \), there exists an \( \omega(\Phi) \geq \omega \) such that \( A \) generates an EH(\( \Phi ; \omega(\Phi) \)) semigroup described by (3).

REMARKS. (1) If \( c > e \), then \( A \) is a finite operator, the power series \( \exp(zA) = \sum ((zA)^n/n!) \) yields an entire analytic continuation of \( \{ T_t : t \in [0, \infty) \} \), and for any \( \omega > r_\Phi(A) \) (the spectral radius) the extension is in EH(\( \pi/2 ; \omega \)).

(2) If the conditions in (a) or (b) hold for all \( 0 < t < \infty \), rather than just for \( 0 < t \leq 1 \), then the formula (3) defines a semigroup extendable by limits to \( S_\Phi \) with \( \Phi = \sin^{-1}(ce^{-1}) \) to yield a semigroup in EH(\( \Phi ; \omega \)) with the same growth rate on \( S_\Phi \) as on \( [0, \infty) \). For \( c = e \) and \( \omega = 0 \), this gives a condition for holomorphic continuations equicontinuous in the closed half-plane. (This remark is due in large part to B. Dembart (private communication).)

(3) The methods of Yosida (loc. cit.) can be used to prove that (a)–(c) are also equivalent to a resolvent condition:

(e) For any \( e > 0 \) there exists a constant \( d_e > 0 \) such that \( [d_e, \lambda + \omega - A]^{-1} : \lambda > 0, n = 0, 1, \ldots \} \) is equicontinuous. Theorem 6 can be used to prove that this in fact is equivalent to the superficially much stronger resolvent condition:

(f) For some calibration \( \Gamma, \Phi > 0 \) and \( \omega(\Phi) \geq \omega, \sigma_\Gamma(A) \subset \Delta_\Phi + \omega(\Phi) \) and if \( d_\lambda = \text{dist}(\lambda, \Delta_\Phi + \omega(\Phi)) \) then \( \| (\lambda - A)^{-1} \| \leq d_\lambda^{-1} \).

6. Integration of certain abstract heat equations. Suppose that \( \{ T_i^\alpha : t \in \mathbb{R} \}, 1 \leq j \leq n \), is a collection of \( n \) smooth \( EC_0(0) \) groups on \( \mathcal{X} \), with continuous generators \( A_j \in \mathcal{L}(\mathcal{X}) \). Then we define an abstract Laplacian operator by

\[
L = \sum \{ A_j^2 : 1 \leq j \leq n \}
\]

in \( \mathcal{L}(\mathcal{X}) \), and seek solutions to the abstract heat equation for \( \{ u(t) : t \in [0, \infty) \} \):

\[
\frac{d}{dt} u(t) = Lu(t), \quad t \in (0, \infty); \quad u(0) = u_0 \in \mathcal{X}.
\]

Utilizing Yosida's ideas on "holomorphic Markov processes" [11], the following partial results have been obtained.

THEOREM 7. Suppose the multiplicative group in \( \mathcal{L}(\mathcal{X}) \) generated by
all of the operators \( \{ T_j^t : t \in \mathbb{R} \text{ and } 1 \leq j \leq n \} \) is equicontinuous. Then \( L \) generates a smooth \( EH(0; 0) \) semigroup \( \{ P_t : t \in [0, \infty) \} \) such that for every initial condition \( u_0 \in \mathcal{X} \), \( u(t) = P_t u_0 \) solves (5).

**Theorem 8.** Suppose \( T_s^T T_t^T = T_s^T T_t^T \) for all \( s, t \in \mathbb{R} \) and \( j, k \) between 1 and \( n \). Then for every \( 0 \leq \Phi \leq \pi/2 \) there exists an \( \omega(\Phi) \) such that \( L \) generates an \( EH(\Phi; \omega(\Phi)) \) semigroup \( \{ P_s : s \in S_\Phi \} \) which is equicontinuous on the real axis and such that \( u(t) = P_t u_0 \) solves (5) as above.

**Remark.** Analytic continuation off the axis in the setting of Theorem 7 is likely, but has not conclusively been established.

**Example.** The Laplacian on the \( n \)-torus discussed in Example 2 of [8] is typical of the generalized Laplacians treated above. (Notice that both theorems apply.) This gives an entirely abstract, operator-theoretic integration of the heat equation on the \( n \)-torus which avoids the “deficiency” problems and ellipticity techniques of the \( L^2 \) theory.

**References**


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