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GENERATION OF EQUICONTINUOUS SEMIGROUPS BY HERMITIAN AND SECTORIAL OPERATORS. II

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1. **Introduction.** This announcement concerns the topological aspects of the generation theory of equicontinuous semigroups and groups of operators on a complete complex locally convex space (lcs) $\mathfrak{X}$, and uses the recalibration theorem from [6] to relate these to the more geometrical aspects treated in [8] (with which the reader is assumed to be familiar). Perturbation techniques from [7], along with other devices, are used to develop applications to the theory of abstract heat equations and to the theory of distribution semigroups. Details will appear in [9].

2. **Quasi-equicontinuous semigroups.** The semigroups considered here generalize the contraction holomorphic semigroups $\mathrm{CH}(\Phi, \Gamma)$ on a complete complex lcs $\mathfrak{X}$ discussed in [8], where for, $0 \leqq \Phi \leqq \pi/2$, $S_\Phi = \{z \in \boldsymbol{C}: |\arg z| \leqq \Phi\}$ and $\Delta_\Phi = \{z \in \boldsymbol{C}: \pi/2 + \Phi \leqq \arg z \leqq 3\pi/2 - \Phi\}$.

Definition 1. Let $\omega \geqq 0$. Then a family $\{T_z : z \in S_\Phi\} \subset \mathcal{L}(\mathfrak{X})$ of continuous linear transformations is a *quasi-equicontinuous holomorphic semigroup of type* ω, or is in $\mathrm{EH}(\Phi; \omega)$ iff

(a) it satisfies the usual algebraic, continuity and holomorphy conditions as in Definition (1a) of [8], and

(b) the family $\{e^{-\omega z} T_z : z \in S_\Phi\}$ is equicontinuous in $\mathcal{L}(\mathfrak{X})$.

Examples. (1) If $\{T_t : t \in [0, \infty)\}$ is a classical C_0 semigroup on a B-space [3], and $\omega > \omega_0 = \lim\{(t^{-1} \log \|T_t\|) : t \to \infty\}$, then $\|T_t\| \leqq M e^{\omega t}$ for suitable M and the semigroup is in $\mathrm{EH}(0; \omega)$ since operator-norm-bounded sets are equicontinuous. Similarly, every semigroup in Hille's class $H(-\Psi, \Psi)$ on a B-space [3] is in $\mathrm{EH}(\Phi; \omega(\Phi))$ for every $\Phi < \Psi$ and suitable $\omega(\Phi)$.

(2) Every $\mathrm{CH}(\Phi, \Gamma)$ semigroup from [8] is in $\mathrm{EH}(\Phi; 0)$.

(3) Every equicontinuous C_0 semigroup as in Yosida [10] is in

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EH(0; 0); Miyadera [5] discusses EH(0; ω) semigroups on Fréchet spaces.

As in Definition 2 of [8], the *(infinitesimal) generator* A of an EH(Φ; ω) semigroup is defined, wherever the limit exists, by

$$(1) \qquad Au = \lim\{t^{-1}(T_t u - u) : t \to 0 \text{ in } [0, \infty] \subset S_\Phi\}$$

and the semigroup is *smooth* iff this limit exists for all $u \in \mathfrak{X}$.

THEOREM 1. *Let* $0 \leqq \Phi \leqq \pi/2$ *and let* $\omega \geqq 0$. *The following conditions on an operator* A *are equivalent.*

(a) *The domain* $D(A)$ *is dense in* $\mathfrak{X}$, A *is closed, the finite spectrum of* A, $\sigma_F(A) \subset \Delta_\Phi + \omega$ *(cf. [6]) and if* $d_\lambda = \operatorname{dist}(\lambda, \Delta_\Phi + \omega)$ *then*

$$\{[d_\lambda(\lambda - A)^{-1}]^n : d_\lambda > 0, n = 0, 1, \cdots\}$$

is equicontinuous.

(b) *The operator* A *is closed and densely defined, and* $A - \omega$ *is* Φ-*sectorial with respect to some calibration* Γ *for* $\mathfrak{X}$.

(c) *There exists a calibration* Γ *for* $\mathfrak{X}$ *such that* $A - \omega$ *generates a semigroup* $\{T_z^\omega = e^{-\omega z} T_z : z \in S_\Phi\}$ *in* CH(Φ, Γ).

(d) *The operator* A *generates an* EH(Φ; ω) *semigroup* $\{T_z : z \in S_\Phi\}$.

PROOF SCHEME. All implications in (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d) are routine from [6], [7], and [8], with the exception of (b)$\Rightarrow$(*a*), which requires recalibration by an equicontinuous simply closed convex balanced semigroup S of operators generated by the $d_\lambda(\lambda - A)^{-1}$ for $\lambda \in S_\Phi$.

Similarly, a group $\{T_t : t \in \boldsymbol{R}\}$ is in $EC_0(\omega)$ iff it is C_0 as in Definition 4 of [8] and $\{e^{-\omega|t|} T_t : t \in \boldsymbol{R}\}$ is equicontinuous. The analog of Theorem 1 is straightforward, with $\sigma_F(A)$ and $W(A, \Lambda_*)$ contained in the strip $\{z : |\operatorname{Re}(z)| \leqq \omega\}$ in (a) and (b) respectively. The case $\omega = 0$ is of primary interest.

THEOREM 2. *If* A *is as in Theorem* 1, *the following are equivalent.*

(a) *The finite spectrum of closed, densely defined* A *is pure imaginary, and* $\{[|\operatorname{Re}(\lambda)|(\lambda - A)^{-1}]^n : n = 0, 1, \cdots, \operatorname{Re}(\lambda) \neq 0\}$ *is equicontinuous.*

(b) *The operator* iA *is closed, densely defined, and hermitian with respect to a suitable calibration* Γ.

(c) A *generates a generalized unitary group* $\{T_t : t \in \boldsymbol{R}\}$ *in* $RC_0(\Gamma)$ *for some suitable* Γ.

(d) A *generates an* $EC_0(0)$ *group* $\{T_t : t \in \boldsymbol{R}\}$.

3. Semigroups on Banach sub-and-super-spaces. We show here that the properties of EH(Φ; ω) semigroups are entirely determined by their natural classical actions on certain appropriate classes of

Banach subspaces and super-spaces of the given lcs $\mathfrak{X}$. To simplify the discussion, suppose that $\mathfrak{X}$ admits a continuous norm, hence a calibration Γ consisting of norms. (Recalibration preserves this property.)

Then the super-spaces are formed by completing $\mathfrak{X}$ with respect to a particular norm $p\in\Gamma$ to obtain a B-space $\mathfrak{X}_p$.

THEOREM 3. *Let* $\{T_z:z\in S_\Phi\}$ *be an operator-valued function. Then it is an* EH$(\Phi;\omega)$ *semigroup iff there exists a norm-calibration* Γ *for* $\mathfrak{X}$ *such that every* T_z *has an extension-by-continuity* $T_z^p\in\mathfrak{B}(\mathfrak{X}_p)$ *and* $\{e^{-\omega z}T_z^p:z\in S_\Phi\}$ *forms a* CH$(\Phi, \{p\})$ *semigroup on* $\mathfrak{X}_p$ *for all* $p\in\Gamma$.

The subspaces are defined in terms of functions $F:\Gamma\to(0, \infty)$, letting $\|u\|_F=\{\sup F(p)^{-1}p(u):p\in\Gamma\}\in[0, \infty)$ and

$$\mathfrak{X}_F = \{u \in \mathfrak{X}:\|u\|_F < \infty\}$$

with *bounded* unit ball $\mathfrak{B}_F=\{u\in\mathfrak{X}:\|u\|_F\leqq 1\}$. Then

$$\mathfrak{X} = \bigcup\{\mathfrak{X}_F:F:\Gamma \to (0, \infty)\} = \bigcup\{\mathfrak{B}_F:F:\Gamma \to (0, \infty)\}.$$

THEOREM 4. *Let* $\{T_z:z\in S_\Phi\}$ *be an operator-valued function. Then it is an* EH$(\Phi; \omega)$ *semigroup iff there exists a norm-calibration* Γ *for* $\mathfrak{X}$ *satisfying the two conditions*:

(1) *every* T_z *leaves every* $\mathfrak{X}_F$ *invariant and restricts to a* $T_z^F\in\mathfrak{B}(\mathfrak{X}_F)$ *with* $\|e^{-\omega z}T_z^F\|_F\leqq 1$, *and*

(2) *for some* $\|\cdot\|_F$*-closed,* Γ*-dense subspace* $\mathfrak{Y}_F\subset\mathfrak{X}_F$ *invariant under the* T_z^F, *the restricted* $\{e^{-\omega z}T_z^F:z\in S_\Phi\}\subset\mathfrak{B}(\mathfrak{Y}_F)$ *is a* CH$(\Phi, \{\|\cdot\|_F\})$ *semigroup.*

REMARK. It is not yet clear whether Γ can be chosen so that $\{e^{-\omega z}T_z^F:z\in S_\Phi\}$ is in CH$(\Phi, \{\|\cdot\|_F\})$ on all of every $\mathfrak{X}_F$, although it is necessarily operator-norm holomorphic there on int(S_Φ).

The inevitable variants of these results are also true for groups.

4. **Exponential distribution semigroups.** Lions [4] has defined, and proved a "Hille-Yosida theorem" for, a generalized class of *exponential distribution semigroups of type* $\leqq\omega$, (or ED$(0;\omega)$ semigroups) on a B-space $\mathfrak{X}$. These are $\mathfrak{B}(\mathfrak{X})$-valued distributions (where $\mathfrak{B}(\mathfrak{X})$ carries the norm topology) which restrict to algebra homomorphisms into $\mathfrak{B}(\mathfrak{X})$ from the convolution algebra $\mathfrak{D}_0$ of C^∞ functions φ compactly supported in $[0, \infty)$, forming $\{T(\varphi):\varphi\in\mathfrak{D}_0\}$. In addition to technical hypotheses governing behavior near 0, one assumes that the map $\varphi\to$"$T(e^{-\omega t}\varphi(t))$" extends to a tempered distribution. These arise classically if $\{T_t:t\in[0, \infty)\}$ is a semigroup homomorphism which is strongly continuous on the *open set* $(0, \infty)$ and satisfies the integra-

bility condition at $0: \int_0^1 \|T_t u\|\,dt < \infty$ for all $u \in X$, as in the Hille-Phillips semigroups of class $(0, A)$ [3]. Then $\{T(\varphi): \varphi \in \mathfrak{D}_0\}$ is defined by integration for $\varphi \in \mathfrak{D}_0$ and $u \in \mathfrak{X}$ by

$$T(\varphi)u = \int_0^\infty \varphi(t) T_t u \, dt. \tag{2}$$

A classical theorem of Feller [1], relating these more general classical semigroups to those in $\mathrm{EH}(0; \omega)$, can be generalized using Theorems 3 and 4 along with results due to Fujiwara [2].

THEOREM 5. *If* $\{T(\varphi): \varphi \in \mathfrak{D}_0\}$ *is in* $\mathrm{ED}(0; \omega)$, *then there exists a dense subspace* $\mathfrak{Y} \subset \mathfrak{X}$, *which is a B-space with respect to a stronger norm* $\|\cdot\|_c \geqq \|\cdot\|$, *and a classical* C_0 *semigroup* $\{T_t: t \in [0, \infty)\}$ *on* $(\mathfrak{Y}, \|\cdot\|_c)$ *such that for all* $u \in \mathfrak{Y}$ *and* $\varphi \in \mathfrak{D}_0$, (2) *holds and characterizes* $\{T(\varphi): \varphi \in D_0\}$ *uniquely.*

REMARKS. (1) Either $\mathfrak{Y}$ is first-category in $\mathfrak{X}$, or $\mathfrak{Y} = \mathfrak{X}$ as topological vector spaces, and in the latter case the $\mathrm{ED}(0; \omega)$ semigroup is exactly the integrated form (2) of $\{T_t: t \in [0, \infty)\}$.

(2) "Holomorphic" exponential distribution semigroups have been defined on sectors S_Φ for $0 < \Phi \leqq \pi/2$ in the work of Fujiwara and others [2], leading naturally to a class of $\mathrm{ED}(\Phi; \omega)$ semigroups to which Theorem 5 easily extends.

GENERALIZATION. If $\mathfrak{B}(\mathfrak{X})$ in the definition of an $\mathrm{ED}(\Phi; \omega)$ semigroup is replaced by $\mathfrak{F}_\Gamma(X)$ on a general locally convex space, the entire Lions theory admits a reasonable generalization to this setting. A satisfactory Γ-independent theory has not yet been obtained.

5. **Yosida's analytic continuation theorem.** The recalibration theorem can be used to clarify, strengthen, and trivially generalize an important theorem of Yosida (cf. §IX.10 of [10]) concerning continuation of an $\mathrm{EH}(0; 0)$ semigroup from $[0, \infty)$ to a suitable S_Φ for $\Phi > 0$ in order to obtain an $\mathrm{EH}(\Phi; \omega)$ semigroup.

THEOREM 6. *Let* $\{T_t: t \in [0, \infty)\}$ *be an* $\mathrm{EH}(0; \omega)$ *semigroup with generator* A. *Then* (a), (b) *and* (c) *are equivalent, and imply* (d).

(a) *For all* $t > 0$, $T_t \mathfrak{X} \subset D(A)$, *and there exists a* $c > 0$ *such that* $\{[ct(A - \omega)e^{-\omega t} T_t]^n: 0 < t \leqq 1 \text{ and } n = 0, 1, \cdots\}$ *is an equicontinuous family of operators on* $\mathfrak{X}$.

(b) *For all* $t > 0$, $T_t \mathfrak{X} \subset D(A)$ *and there exists a* $c > 0$ *and a calibration* Γ *for* $\mathfrak{X}$ *such that* $A T_t \in \mathfrak{F}_\Gamma(\mathfrak{X})$ *and* $\|(A - \omega) T_t\|_\Gamma \leqq (cte^{-\omega t})^{-1}$ *for all* $0 < t \leqq 1$.

(c) *For some calibration* Γ *and* $c > 0$ *the series in the formula*

$$(3)\qquad T_z = e^{\omega z} \sum \left\{ \frac{(z-t)^n}{n!} [(A-\omega)e^{-\omega t}T_t]^n : n = 0, 1, \cdots \right\}$$

converges in the $\|\cdot\|_\Gamma$ *sense whenever* $|z-t| < cte^{-1}$.

(d) *For some constant* $c>0$ *and any* $\Phi<\sin^{-1}(ce^{-1})$, *there exists an* $\omega(\Phi)\geqq\omega$ *such that* A *generates an* $\mathrm{EH}(\Phi;\,\omega(\Phi))$ *semigroup described by* (3).

REMARKS. (1) If $c>e$, then A is a finite operator, the power series $\exp(zA)=\sum((zA)^n/n!)$ yields an entire analytic continuation of $\{T_t : t\in[0,\infty)\}$, and for any $\omega>r_E(A)$ (the spectral radius) the extension is in $\mathrm{EH}(\pi/2;\,\omega)$.

(2) If the conditions in (a) or (b) hold for all $0<t<\infty$, rather than just for $0<t\leqq 1$, then the formula (3) defines a semigroup extendable by limits to S_Φ with $\Phi=\sin^{-1}(ce^{-1})$ to yield a semigroup in $\mathrm{EH}(\Phi;\,\omega)$ with the same growth rate on S_Φ as on $[0,\infty)$. For $c=e$ and $\omega=0$, this gives a condition for holomorphic continuations equicontinuous in the closed half-plane. (This remark is due in large part to B. Dembart (private communication).)

(3) The methods of Yosida (loc. cit.) can be used to prove that (a)–(c) are also equivalent to a resolvent condition:

(e) For any $\epsilon>0$ there exists a constant $d_\epsilon>0$ such that $[d_\epsilon\lambda(\lambda+\omega-A)^{-1}]^n : \lambda>0,\ n=0,1,\cdots\}$ is equicontinuous. Theorem 6 can be used to prove that this in fact is equivalent to the superficially much stronger resolvent condition:

(f) For some calibration Γ, $\Phi>0$ and $\omega(\Phi)\geqq\omega$, $\sigma_\Gamma(A)\subset\Delta_\Phi+\omega(\Phi)$ and if $d_\lambda=\operatorname{dist}(\lambda,\Delta_\Phi+\omega(\Phi))$ then $\|(\lambda-A)^{-1}\|\leqq d_\lambda^{-1}$.

6. **Integration of certain abstract heat equations.** Suppose that $\{T_t^{(j)} : t\in R\}$, $1\leqq j\leqq n$, is a collection of n smooth EC_0 (0) groups on $\mathfrak{X}$, with continuous generators $A_j\in\mathfrak{L}(\mathfrak{X})$. Then we define an *abstract Laplacian operator* by

$$(4)\qquad L = \sum\{A_j^2 : 1 \leqq j \leqq n\}$$

in $\mathfrak{L}(\mathfrak{X})$, and seek solutions to the *abstract heat equation* for $\{u(t) : t\in[0,\infty)\}$:

$$(5)\qquad \frac{d}{dt}u(t) = Lu(t), \qquad t\in(0,\infty);\quad u(0)=u_0\in\mathfrak{X}.$$

Utilizing Yosida's ideas on "holomorphic Markov processes" [11], the following partial results have been obtained.

THEOREM 7. *Suppose the multiplicative group in* $\mathfrak{L}(\mathfrak{X})$ *generated by* *all of the operators* $\{T_t^{(j)}: t\in \boldsymbol{R}$ *and* $1\leqq j\leqq n\}$ *is equicontinuous. Then* L *generates a smooth* EH(0; 0) *semigroup* $\{P_t: t\in[0, \infty)\}$ *such that for every initial condition* $u_0\in\mathfrak{X}$, $u(t)=P_tu_0$ *solves* (5).

THEOREM 8. *Suppose* $T_s^{(j)}T_t^{(k)}=T_t^{(k)}T_s^{(j)}$ *for all* s, $t\in\boldsymbol{R}$ *and* j, k *between* 1 *and* n. *Then for every* $0\leqq\Phi\leqq\pi/2$ *there exists an* $\omega(\Phi)$ *such that* L *generates an* EH(Φ; $\omega(\Phi)$) *semigroup* $\{P_z: z\in S_\Phi\}$ *which is equicontinuous on the real axis and such that* $u(t)=P_tu_0$ *solves* (5) *as above.*

REMARK. Analytic continuation off the axis in the setting of Theorem 7 is likely, but has not conclusively been established.

EXAMPLE. The Laplacian on the n-torus discussed in Example 2 of [8] is typical of the generalized Laplacians treated above. (Notice that both theorems apply.) This gives an entirely abstract, operator-theoretic integration of the heat equation on the n-torus which avoids the "deficiency" problems and ellipticity techniques of the L^2 theory.

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