MULTIPLICATIVE OPERATOR FUNCTIONALS
OF A MARKOV PROCESS

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1. Introduction. Let $X = \langle x(t), \xi, \mathcal{M}_t, P_x \rangle$ be a right continuous Markov process on a state space $(E, \mathcal{B})$. Let $L$ be a fixed Banach space. A multiplicative operator functional (MOF) of $(X, L)$ is a mapping $(t, \omega) \mapsto M(t, \omega)$ of $[0, \infty) \times \Omega$ to bounded operators on $L$ which possesses the following properties:

(1a) $\omega \mapsto M(t, \omega)f$ is $\mathcal{M}_t$ measurable for each $t \geq 0, f \in L$.

(1b) $t \mapsto M(t, \omega)f$ is right continuous a.s. for each $f \in L$.

(1c) $M(t+s, \omega)f = M(t, \omega)M(s, \theta_\omega)f$ a.s. for each $s, t \geq 0, f \in L$.

(1d) $M(0, \omega)$ is the identity operator on $L$.

If $M$ is a multiplicative operator functional of $(X, L)$ the expectation semigroup is defined on the direct sum Banach space $L = \bigoplus E L$ by the equation

$$
(\hat{T}(t)f)_x = E_x[M(t, \omega)f_x(t, \omega)].
$$

The MOF concept has appeared in several places recently. We were led to the idea by the work of Griego and Hersh [4], [5] who constructed examples of an MOF when $X$ is a Markov chain with a finite number of states and $L$ is arbitrary. Here $M(t, \omega)$ is a finite random product of semigroups. Earlier Babbitt [1] had studied the case $X =$ Wiener process on $\mathbb{R}^n$, $L =$ finite-dimensional vector space. In this case $M(t)$ is a solution to a system of Itô stochastic differential equations. If $X$ is a Poisson process and $L$ is a Banach space of continuous functions on $\mathbb{R}^1$, we can specialize the MOF concept to represent the semigroups studied by Çinlar and Pinsky in a problem in storage theory. In this case the infinitesimal operator of the associated semigroup is an integro-differential operator. Further applications will be discussed in another publication.

2. Main results. Here we will give the notations and state the main results. Proofs will not be given. Detailed proofs will appear elsewhere.

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If $L$ is a Banach space and $E$ is an arbitrary index set, $L$ is the Banach space of all tuples $\bar{f} = (f_x)_{x \in E}$ where $f_x \in L$ for each $x$; $L$ is equipped with componentwise addition and scalar multiplication; we use the norm $\|\bar{f}\|_L = \sup |f_x|$. A contractive MOF is an MOF with the additional property that $\|M(t, \omega)f\| \leq \|f\|$ a.s. for $t \geq 0, f \in L$.

If $X$ has no instantaneous states, let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ be the jump times of $x(t)$ and let $N(t)$ be the number of jumps in $[0, t]$.

**Theorem 1.** Let $E = \{1, 2, \cdots, N\}$ and let $X$ be a finite Markov chain on $E$ with $\xi = +\infty$. Let $M$ be a contractive multiplicative operator functional of $(X, L)$. Then there exist strongly continuous contraction semigroups $T^1(t), \cdots, T_N(t)$ on $L$ and bounded operators $\{\Pi_{\alpha}\}_{1 \leq \alpha \leq N}$ on $L$ such that for $t \geq 0, f \in L$,

$$M(t)f = \prod_{i=1}^{N(t)} (\tau_i - \tau_{i-1}) \Pi_{\tau_{\tau_{t-1}}(\tau_i)} T_{\tau_{\tau_{t-1}}(\tau_i)}(t - \tau_N(t))f.$$  

The key step in the proof of this result is the observation that $M(t, \omega)$ is a semigroup up until the first jump time of $X$. More precisely we have a

**Lemma.** If $0 \leq s, t$ and $t + s < \tau_1(\omega)$, then

$$M(t + s, \omega) = M(t, \omega)M(s, \omega) \text{ a.s.}$$

In the case of a continuous MOF we can specialize the form (2.1) and also relax the hypothesis.

**Theorem 2.** Let $E = \{1, 2, \cdots, N\}$ and let $X$ be a finite Markov chain on $E$ with $\xi = +\infty$. Let $M$ be a multiplicative operator functional of $(X, L)$ such that the mapping $t \mapsto M(t, \omega)f$ is continuous from $[0, \infty)$ to $L$ for each $f \in L, \omega \in \Omega$. Then there exist strongly continuous semigroups $T^1(t), \cdots, T_N(t)$ on $L$ such that, for $f \in L$,

$$M(t)f = T_{\tau_{\tau_{t-1}}(\tau_1)}(t)T_{\tau_{\tau_{t-1}}(\tau_2)}(t - \tau_1) \cdots T_{\tau_{\tau_{t-1}}(\tau_N(t))}(t - \tau_N(t))f.$$  

**Remark.** The form (2.2) is termed a random evolution by Griego and Hersh. In some sense $M(t)$ evolves by the random equation $dM/dt = M(t)A_x(t)$ where $A_x$ is the infinitesimal operator of the semigroup $T_x(t)$.

**3. Examples and other applications.** We limit this discussion to two classes of examples of MOF’s. Other applications will be discussed elsewhere.

I. Let $X = (x(t), +\infty, \mathcal{M}_t, P_x)$ be a separable Wiener process on $R^n$. Let $B_1, \cdots, B_n$ and $V$ be bounded continuous complex $N \times N$
matrix-valued functions on $R^n$ and let $D(B; V)$ be the closure in $L^2(R^n; C^N)$ of the differential operator

$$D(B; V) = \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} \right)^2 + \sum_{j=1}^{n} B_j(x) \frac{\partial}{\partial x_j} + V(x)$$

with domain $C_0^\infty(R^n; C^N)$. Babbitt [1] has shown that there exists a complex $N \times N$ matrix-valued function $\alpha(t, \omega) = \alpha$ on $\Omega \times [0, \infty)$ such that $\alpha$ is adapted to $\mathcal{F}_t$, the map $t \rightarrow \alpha(t, \omega)$ is continuous on $[0, \infty)$ a.s., and $\alpha$ is a solution of the stochastic integral equation

$$\alpha(t, \omega) = I + \sum_{j=1}^{n} \int_{0}^{t} \alpha(\tau, \omega) B_j(x(\tau)) \, dx_j(\tau) + \int_{0}^{t} \alpha(\tau, \omega) V(x(\tau)) \, d\tau$$

a.s. with respect to $P_x$; here $I$ is the identity matrix and

$$\sum_{j=1}^{n} \int_{0}^{t} \alpha(\tau, \omega) B_j(x(\tau)) \, dx_j(\tau)$$

is an Itô stochastic integral. Furthermore $\alpha$ satisfies the bound $E_x \{ |\alpha(t)|^2 \} \leq Q_T$ for some constant $Q_T$ independent of $x$, for $0 \leq t \leq T$.

Now we note that $\alpha(t+, \omega)$ and $\alpha(t, \omega) \alpha(t, \theta \omega)$ (juxtaposition means matrix multiplication) both solve the same stochastic integral equation. Thus by the uniqueness theorem for stochastic integral equations (see [6, p. 54 ff.]) they must be equal a.e. Thus $\alpha(t, \omega)$ satisfies the multiplicative law $\alpha(t+s, \omega) = \alpha(t, \omega) \alpha(s, \theta \omega)$. The formula

$$T_t\phi(x) = E_x[\alpha(t, \omega)\phi(x(t))]$$

according to Babbitt defines a semigroup on $L^2(R^n; C)$ whose infinitesimal operator is $D(B; V)$. Conversely, if $\alpha(t, \omega)$ is a matrix which satisfies the above multiplicative law together with certain smoothness conditions, then $\alpha$ satisfies a stochastic integral equation of the above type, as we shall show in another publication.

II. As a second example, let $X$ be a Poisson process with rate $a (0 < a < \infty)$ with jump times $0 < \tau_1 < \tau_2 < \cdots < \infty$. Let $L$ be the Banach space of all continuous functions on $[0, \infty)$ which vanish at $\infty$. Let $T(\cdot)$ be the group on $L$ with $T(\tau)f(x) = f(k_\tau(x))$ where $k_\tau(x)$ is a solution of the o.d.e $k_\tau' = -r(k_\tau)$, $k_\tau(0) = x$. Here $r$ is a strictly increasing differentiable function with $r(0) = 0$. Let $\Pi$ be the convolution operator on $L$ given by $\Pi f(x) = \int_{0}^{\infty} f(x-y) \nu(dy)$, where $\nu$ is a fixed probability measure on $[0, \infty)$. Then the formula

$$M(t) = T(\tau_1) \Pi T(\tau_2 - \tau_1) \Pi \cdots \Pi T(t - \tau_N(t))$$

defines a multiplicative operator functional on $(X, L)$ with the
property that the expectation semigroup leaves invariant the class
\( \{ f = (f, f, \cdots) : f \in L \} \). A calculation shows that the infinitesimal
operator of the expectation semigroup is given by an extension of the
operator
\[
A f = -r(x) \frac{\partial f}{\partial x} + a \int_0^\infty (f(x + y) - f(x)) \nu(dy)
\]
acting on smooth functions. This semigroup and its limit as \( t \to \infty \)
were considered in detail in [2].

The author recently became aware of the work of D. W. Stroock
(Comm. Pure Appl. Math. 23 (1970), 447–457). This paper extends
the work of Babbitt to include, in particular, coefficients \((B_j, V)\)
which depend on \( t \) and are merely bounded and measurable.

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