

**L^p BOUNDEDNESS OF CERTAIN
 CONVOLUTION OPERATORS**

BY E. M. STEIN

Communicated by M. H. Protter, December 14, 1970

We announce here several results dealing with the boundedness of convolution operators on $L^p(\mathbb{R}^n)$. These results make use of the idea that boundedness on H^1 often holds when weak type $(1, 1)$ estimates apply (see [5, Chapter VII]), and the recent discovery of Fefferman [3] characterizing the dual of H^1 .

Our first result states in effect that the complex intermediate spaces between $H^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$ are the $L^p(\mathbb{R}^n)$. We state this precisely in a particular but useful case. Let $z \rightarrow T_z$ be a mapping from the closed strip, $0 \leq R(z) \leq 1$, to bounded operators on $L^2(\mathbb{R}^n)$, which is assumed analytic in the interior and strongly continuous and uniformly bounded in the closed strip.

THEOREM 1. *Suppose*

- (1) $\sup_{-\infty < \nu < \infty} \|T_{i\nu}(f)\|_{H^1} \leq M_0 \|f\|_{H^1}, \quad f \in H^1 \cap L^2,$
 (2) $\sup_{-\infty < \nu < \infty} \|T_{1+i\nu}(f)\|_{L^2} \leq M_1 \|f\|_{L^2}, \quad f \in L^2.$

Then $\|T_t(f)\|_p \leq M_t \|f\|_p, f \in L^p \cap L^2, \text{ if } 0 < t < 1, 1/p = 1 - t/2.$

This theorem allows one to obtain certain sharp estimates which did not fall under the scope of previous methods. We give two examples.

First, let $K(x)$ be a distribution of compact support, locally integrable away from the origin, and whose Fourier transform $\hat{K}(x)$ is a function. Assume (following Fefferman [2]) that for some $\theta, 0 \leq \theta < 1$,

- (i) $|\hat{K}(x)| \leq A(1 + |x|)^{-n\theta/2},$
 (ii) $\int_{|x| \geq 2|y|}^{1-\theta} |K(x-y) - K(x)| dx \leq A, \quad |y| \leq 1.$

THEOREM 2. $|x|^\gamma \hat{K}(x)$ is a bounded multiplier for $(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ if $|(1/p) - \frac{1}{2}| = \frac{1}{2} - \gamma/n\theta, \gamma > 0.$

An instance of the above arises with $|x|^\gamma \hat{K}(x) = \theta(x) |x|^{-\beta} \exp i|x|^\alpha,$

AMS 1969 subject classifications. Primary 3067, 4258, 4425, 3538.

Key words and phrases. H^p spaces, L^p spaces, interpolation theorems, convolution operators.

where θ vanishes near the origin, is C^∞ , and identically one for large x , $0 < \alpha < 1$, and $\beta = (n\alpha/2) - \gamma$.

THEOREM 3. *Suppose $d\mu$ is a finite Borel measure whose Fourier transform $\hat{\mu}(x)$ is $O(|x|^{-\delta})$, $|x| \rightarrow \infty$, $\delta > 0$. Then $|x|^\gamma \hat{\mu}(x)$ is a multiplier for $(L^p(\mathbf{R}^n), L^p(\mathbf{R}^n))$, if $|(1/p) - \frac{1}{2}| = \frac{1}{2} - \gamma/2\delta$.*

As a corollary of the theorem we can take σ to be the uniform mass distributed on the surface of the unit sphere in \mathbf{R}^n . Then $((\partial/\partial x)^\alpha \sigma)^\wedge$ is a multiplier for $(L^p(\mathbf{R}^n), L^p(\mathbf{R}^n))$ if $|(1/p) - \frac{1}{2}| = \frac{1}{2} - |\alpha|/(n-1) \geq 0$. This corollary leads to certain new estimates for the wave equation, and it applies also to the problem of spectral synthesis for the algebra $A_p(\mathbf{R}^n)$. (For the latter, see Eymard [1].)

We state also a further property of H^1 which can be viewed as a sort of converse to Theorem 4 in [3]. Again we state only a special case of the result. Let φ be a C^1 function on \mathbf{R}^n of compact support and set $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$. Let $f^*(x) = \sup_{\epsilon > 0, |t| \leq \epsilon} |(f * \varphi_\epsilon)(x-t)|$.

THEOREM 4. *If $f \in H^1$ then $f^* \in L^1$ and*

$$\|f^*\|_{L^1} \leq A\|f\|_{H^1}.$$

Detailed proofs and extensions of these results to H^p , $p < 1$, will appear in [4].

REFERENCES

1. P. Eymard, *Algèbres A_p et convoluteurs de L^p* , Séminaire Bourbaki, #367, Hermann, Paris, 1969/70.
2. C. L. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), 9-36.
3. ———, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. 77 (1971), (to appear).
4. C. L. Fefferman and E. M. Stein (in prep.).
5. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton, 1970.

PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540