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THE SUBGROUP STRUCTURE OF THE HIGMAN-SIMS SIMPLE GROUP

BY SPYROS S. MAGLIVERAS

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The geometry of the Higman-Sims graph is related to the lattice of subgroups of the Higman-Sims simple group, and the maximal subgroups of the group are determined up to conjugacy. Table 3 exhibits the maximal subgroups.

The study is developed in three stages. The existence of several large subgroups is established in Stage I. Stage II deals with the local subgroups, and in Stage III the nonlocal maximal subgroups are found.

The Higman-Sims graph of 100 vertices is denoted by $\mathfrak{S}$, the group of automorphisms of $\mathfrak{S}$ by $\overline{G}$ and the Higman-Sims group by G. Ω denotes the set of vertices of $\mathfrak{S}$.

If $x\in\Omega$, and r is a nonnegative integer, the *circle of radius r about x* is the set $S_r(x)=\{y\in\Omega:d(x, y)=r\}$.

If $H\leqq G$ and H has the orbits $\Delta_1, \Delta_2, \cdots, \Delta_l$ on Ω, for $x\in\Delta_i$, we put $a_H(i, j)=|S_1(x)\cap\Delta_j|$. The matrix $A_H=(a_H(i, j))$ is called the *matrix belonging to H*.

Let $M=(m_{i,j})$ be an $n\times n$ matrix with nonnegative integral entries and constant row-sums. The *domain* of M, $\mathfrak{D}(M)$, is the collection of all partitions $P=\{\Delta_i\}_{i=1}^k$ of $\{1, 2, \cdots, n\}$ such that for $i, j\in\{1, 2, \cdots, k\}$, $x, y\in\Delta_i$ implies that

$$\sum_{q\in\Delta_j} m_{x,q} = \sum_{q\in\Delta_j} m_{y,q} = \overline{m_{i,j}}.$$

We set $M(P)=(\overline{m_{i,j}})$. If $N=M(P)$ for some $P\in\mathfrak{D}(M)$, we say that N *covers* M and write $M\leqq N$. It follows that if H, K are subgroups of G, such that $H\leqq K$, then $A_H\leqq A_K$.

In addition to group-theoretic and permutation methods in the study of the subgroups of G, character-theoretic and combinatorial methods are used in obtaining the desired results. The general combinatorial method of *matrices belonging to subgroups* is a strong technique which can be adapted to a large number of similar investigations.

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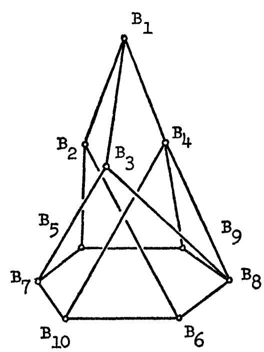
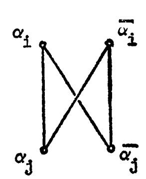
It is shown that the group has a unique representation of degree 100 which is primitive, the point stabilizer being isomorphic to M_{22}. G has two primitive inequivalent representations of degree 176 with point stabilizers isomorphic to $P\Sigma U_3(5^2)$. These correspond to the two conjugacy classes of A_7's in M_{22}. The two conjugacy classes of $P\Sigma U_3(5^2)$ are fused in $\overline{G}$. The group has two primitive inequivalent representations of degree 1100, one on the set of *edges* of $\mathfrak{S}$ with object stabilizer $G_{(\dot{2})}=N_G(M_{21})$ and the other with object stabilizer isomorphic to S_8. S_8 is also the set stabilizer in G of the fix of an involution z_3 in $\overline{G}-G$. The group acts primitively on both classes of involutions 2_1, 2_2 of G. $C_G(2_1)$ is a split extension of a group $K=K(2_1)$, of known structure and of order 64, by S_5. $C_G(2_2)$ is isomorphic to $\langle 2_2\rangle \times P\Gamma L_2(9)$. The set stabilizer of a *nonedge* of $\mathfrak{S}$, $G_{(\ddot{2})}$, is maximal in G, is isomorphic to a split extension of an elementary abelian group of order 2^4 by S_6, and contains the holomorph of 2^4.

If $z\in 2_1$ then $|\text{fix } z|=20$ and $C_G(z)$ acts imprimitively with block length 2 on fix z. The ten blocks $\{B_i\}_{i=1}^{10}$, $B_i=\{\alpha_i, \bar{\alpha}_i\}$ have incidence



where  if and only if 

The three blocks, B_2, B_3, B_4, joined to a given block B_1, form the "tripair" associated with z and $B_1.z$ is recoverable from any one of its ten tripairs. In the canonical form:

$$B_1;\quad B_2, B_3, B_4;\quad B_5, B_6;\quad B_7, B_8;\quad B_9, B_{10},$$

the 15 permutations of type 2^3 on $\Delta_6 = B_2 \cup B_3 \cup B_4$ form the 15 tripairs of the 15 involutions of the elementary abelian group of order 2^4 in $G_{(B_1)} = G_{(\ddot{2})}$. The complete 2^4 in $G_{(\ddot{2})}$ is recoverable from the block B_1 and the graph, and via the elementary abelian groups of order 2^4 the involution z can be recovered in 10 different ways, one for each block.

TABLE 1

Name	*Type*	$C_G(V_4)$	*Fixed Points*	β
V_a	$2_1 2_1 2_1$	2^8	12	$11/2^8 \cdot 3$
V_b	$2_1 2_1 2_1$	$2^5 \cdot 3$	8	
V_c	$2_1 2_2 2_2$	2^6	0	$5/2^6 \cdot 3$
V_d	$2_1 2_2 2_2$	$2^5 \cdot 3$	0	
V_e	$2_2 2_2 2_2$	$2^4 \cdot 5$	0	$1/2^3 \cdot 5$

There are five conjugacy classes of Klein groups in G as indicated in Table 1.

If $\mathfrak{B}$ is the collection of all elementary abelian 2-groups containing a V_b, $\mathfrak{A}$ the collection of all elementary abelian 2-groups whose V_4's are all of type V_a, and if $\mathfrak{J}$ are those elementary abelian 2-groups which contain an involution of 2_2, then $\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{J}$ exhausts the elementary abelian 2-groups in G and $\mathfrak{A} \cap \mathfrak{B} = \mathfrak{A} \cap \mathfrak{J} = \varnothing$. Table 2 exhibits these 2-groups. The fusion in G of all elementary abelian 2-groups has been determined except for the case of the elementary abelian groups of order 8 in $\mathfrak{J}$. The latter can not yield normalizers which are maximal in G.

Besides $C_G(2_1)$ and $C_G(2_2)$, there is, up to conjugacy, one other maximal subgroup of G which is the normalizer of a 2-group. This is the normalizer of $V_{a,8}$ of order $2^9 \cdot 3 \cdot 7$.

There is exactly one more local subgroup which is maximal. This is the normalizer of the cyclic group generated by the element of order five whose centralizer is of order 300 in G. This group N, of order 1200, is imprimitive of type 20^5 on Ω and is also the normalizer of one of the A_5's in G.

In addition to the maximal subgroups mentioned above it is shown that there are two classes of M_{11}'s in G which are obtainable as

TABLE 2

Family	Name	Order	Type	Centralizer	Normalizer
—	2_1	2	—	$C_G(2_1)$	$C_G(2_1)$ maximal
—	2_2	2	—	$C_G(2_2)$	$C_G(2_2)$ maximal
$\mathfrak{B}$	V_b	4	pure-2_1	$S_4 \times V_b$	$S_4 \times S_4 \leqq S_8$
$\mathfrak{B}$	$V_{b,8}$	8	pure-2_1	$D_4 \times V_b$	$\leqq G_{(\ddot{2})}$
$\mathfrak{B}, \mathfrak{J}$	$V_{b,8'}$	8	mixed	$V_c \times V_b = V_{b,16'}$	$\leqq S_8$
$\mathfrak{B}$	$V_{b,16}$	16	pure-2_1	$V_b \times V_b = V_{b,16}$	$\leqq G_{(\ddot{2})}$
$\mathfrak{B}, \mathfrak{J}$	$V_{b,16'}$	16	mixed	$V_{b,16'}$	$\leqq G_{(\ddot{2})}$
$\mathfrak{A}$	V_a	4	pure-2_1	2^8	$\leqq N(V_{a,8})$
$\mathfrak{A}$	$V_{a,8}$	8	pure-2_1	2^6	$2^3 \backslash 2^3 \backslash PSL_3(2)$ maximal
$\mathfrak{J}$	V_c	4	$2_1 2_2 2_2$	2^6	$\leqq C_G(2_1)$
$\mathfrak{J}$	V_d	4	$2_1 2_2 2_2$	$2^5 \cdot 3$	$\leqq C_G(2_1)$
$\mathfrak{J}$	V_e	4	pure-2_2	$2^4 \cdot 3$	$\leqq C_G(2_2)$
$\mathfrak{J}$	k orbits, $2 \leqq k \leqq 4$	8	mixed	—	not maximal

stabilizers of certain null subgraphs of 12 points in $\mathfrak{S}$. It is further shown that the above set is the complete set of conjugacy classes of maximal subgroups of G.

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TABLE 3

Subgroup	Order	Orbit Type
1. $G_{[1]} \cong M_{22}$	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	[1, 22, 77]
2. $G_{(\dot{2})} \cong \overline{M}_{21}$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[2, 42, 56]
3. $G_{(\ddot{2})} \cong 2^4 \backslash\backslash S_6$	$2^8 \cdot 3^2 \cdot 5$	[2, 6, 32, 60]
4. $G_{(\dot{30})} \cong S_8$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[30, 70]
5. $P\Sigma U_3(5^2)$	$2^5 \cdot 3^2 \cdot 5^3 \cdot 7$	*imprimitive*: 50^2
6. $P\Sigma U_3(5^2)'$	$2^5 \cdot 3^2 \cdot 5^3 \cdot 7$	*imprimitive*: 50^2
7. $C_G(2_1) = G_{(\dot{20})} \cong 2^6 \backslash\backslash S_5$	$2^9 \cdot 3 \cdot 5$	[20, 80]
8. $C_G(2_2) \cong C_2 \times P\Gamma L_2(9)$	$2^6 \cdot 3^2 \cdot 5$	[40, 60]
9. $N_G(V_{a,8}) \cong 2^3 \backslash 2^3 \backslash PSL_3(2)$	$2^9 \cdot 3 \cdot 7$	[8, 28, 64]
10. $N_G\langle 5_2 \rangle = N_G(A_{5_3})$	$2^4 \cdot 3 \cdot 5^2$	*imprimitive*: 20^5
11. $G_{(\dot{12})} \cong M_{11}$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[12, 22, 66]
12. $G_{(\ddot{12})} \cong M_{11}$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	[12, 22, 66]

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State University of New York, Oswego, New York 13126