A NONSTANDARD REPRESENTATION OF MEASURABLE SPACES AND $L_\infty$

BY PETER A. LOEB

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The results given in this note were obtained by applying to measure theory the methods of nonstandard analysis developed by Abraham Robinson [5]. Amplifications of these results with proofs will be published elsewhere. It is shown here that there are linear mappings from an arbitrary, real $L_\infty$ space and its dual $L_\infty^*$ into Euclidean $\omega$-space $E^\omega$, where $\omega$ is an infinite integer. Finite valued, finitely additive measures on the underlying measurable space are also mapped onto elements of $E^\omega$, and integrals are infinitesimally close to the corresponding inner products in $E^\omega$. Yosida and Hewitt’s representation of $L_\infty^*$ [6] is an immediate consequence of these results.

In general, we use Robinson’s notation [5]. If we have an enlargement of a structure that contains the set $R$ of real numbers, then $*R$ denotes the set of nonstandard real numbers and $*N$, the set of nonstandard natural numbers. A set $S$ is called *finite if there is an internal bijection from an initial segment of $*N$ onto $S$; a *finite set has all of the “formal” properties of a finite set. Given $b$ and $c$ in $*R$, we write $b \sim c$ if $b - c$ is in the monad of 0; when $b$ is finite, we write $b^*$ for the unique, standard real number in the monad of $b$.

1. The partition $P$ and bounded measurable functions. Let $X$ be an infinite set and $\mathcal{M}$ an infinite $\sigma$-algebra of subsets of $X$. Fix an enlargement of a structure that contains $X$, $\mathcal{M}$, and the extended real numbers. There is a *finite, $*\mathcal{M}$-measurable partition $P$ of $*X$ such that $P$ is finer than any finite $\mathcal{M}$-measurable partition of $X$. That is, $P \subset *\mathcal{M}$ has the following properties:

(i) There is an infinite integer $\omega_P \in *N$ and an internal bijection from $I = \{i \in *N: 1 \leq i \leq \omega_P\}$ onto $P$. Thus we may write $P = \{A_i: i \in I\}$.

(ii) If $i$ and $j$ are in $I$ and $i \neq j$, then $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$.


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2 These results were announced at the 1970 Oberwolfach conference on nonstandard analysis.
(iii) \( \star X = \bigcup_{i \in I} A_i \).

(iv) For each \( B \subseteq \mathcal{M} \), let \( I_B = \{ i \in I : A_i \subseteq \star B \} \). Then \( I_B \) is \( \star \) finite, and 
\( \star B = \bigcup_{i \in I_B} A_i \).

(v) Let \( M \) be the set of \( \mathcal{M} \)-measurable functions on \( X \), and \( MB \), the set of bounded functions in \( M \). For each \( f \in MB \) and \( i \in I \), 
\( \sup_{x \in A_i} *f(x) - \inf_{x \in A_i} *f(x) \approx 0 \).

Given the partition \( P \), we let \( E \) denote the set of all internal mappings from \( I \) into \( \star R \). The set \( E \) has all of the “formal” properties of Euclidean \( n \)-space. We shall write \( x_i \) instead of \( x(i) \) for \( x \in E \) and \( i \in I \), and we shall write \( x \equiv y \) if \( x, y \in E \) and \( x \cap y = \emptyset \), \( \forall i \in I \). Let \( c_P \) denote a fixed internal choice function defined on \( I \) with \( c_P(i) \in A_i \) \( \forall i \in I \). Let \( T \) denote the mapping from \( MB \) into \( E \) defined by setting \( T(f)(i) = *f(c_P(i)) \) for each \( f \in MB \) and \( i \in I \).

**Proposition 1.** Given \( f, g \) in \( MB \) and \( \alpha, \beta \) in \( R \), 
\( T(\alpha f + \beta g) = \alpha T(f) + \beta T(g) \) and \( T(f) \neq T(g) \) iff \( f \neq g \).

2. Measures and integration. Let \( \Phi(X, \mathcal{M}) \), or simply \( \Phi \), denote the set of all finitely additive real-valued functions \( \mu \) on \( \mathcal{M} \) such that 
\( \sup_{B \in \mathcal{M}} | \mu(B) | < +\infty \). Let \( U \) be the mapping of \( \Phi \) into \( E \) defined by setting 
\( U(\mu)(i) = *\mu(A_i) \) for each \( \mu \in \Phi \) and \( i \in I \). Clearly, \( U \) preserves addition and multiplication by real numbers. Conversely, if \( e \in E \) and both \( \sum_{i \in I} (e_i \cup 0) \) and \( \sum_{i \in I} (-e_i \cup 0) \) are finite in \( \star R \), let \( \varphi(e) \) be that element of \( \Phi \) such that for each \( B \in \mathcal{M}, \varphi(e)(B) = \sigma \sum_{i \in I} e_i \). (Note that we are writing \( \sum \) instead of \( \star \sum \) for the extension of the summation operator.) For each \( \mu \in \Phi \), \( \varphi(U(\mu)) = \mu \), but in general, 
\( U(\varphi(e)) \neq e \). If \( \mu \) and \( \nu \) are in \( \Phi \), then \( U(\mu) \cap U(\nu) \approx U(\mu \cap \nu) \), and 
\( \sigma \sum_{i \in I} | U(\mu)(i) | = | \mu | (X) \).

Let \( \Phi_e \) and \( \Phi_p \) be, respectively, the set of countably additive and the set of purely finitely additive elements of \( \Phi \). Yosida and Hewitt’s Theorem 1.19 [6] has the following extension:

**Theorem 1.** There is a set \( K \subseteq \star \mathcal{M} \) such that for all \( \mu \in \Phi_e, | \mu | (K) \approx 0 \) and for all \( \nu \in \Phi_p, | \nu | (\star X - K) = 0 \).

Without loss of generality, we assume that \( K = U \{ A_i \in P : A_i \subseteq K \} \).

If \( \mu = \mu_e + \mu_p \) is the decomposition of an element \( \mu \) in \( \Phi = \Phi_e \oplus \Phi_p \), then 
when \( A_i \subseteq \star X - K \), \( U(\mu)(i) = U(\mu_e)(i) \) and when \( A_i \subseteq K \), \( U(\mu)(i) \approx U(\mu_p)(i) \). We next show that there is a “maximum” null set for each \( \mu \in \Phi^+ \), and we extend the Hahn decomposition theorem for countably additive signed measures.

**Theorem 2.** Let \( \mu \) be an arbitrary, finitely additive signed measure on \( (X, \mathcal{M}) \). Let

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\[ A_+ = \bigcup \{ A_i \in P : \mu(A_i) > 0 \}, \quad A_- = \bigcup \{ A_i \in P : \mu(A_i) < 0 \}, \]
and
\[ A_0 = \bigcup \{ A_i \in P : \mu(A_i) = 0 \}. \]

Then \( \mu(A_0) = 0 \), and for each \( \mu \)-null set \( B \subseteq \mathcal{M} \), \( B \subseteq A_0 \). If there exists a \( \mu \)-positive set \( B_+ \) and a \( \mu \)-negative set \( B_- \) in \( \mathcal{M} \) with \( X = B_+ \cup B_- \) and \( B_+ \cap B_- = \emptyset \), then \( A_+ \subseteq \mathcal{B}_+ \), \( A_- \subseteq \mathcal{B}_- \), and each \( A_i \in P \) is either a \( \mu \)-positive set or a \( \mu \)-negative set.

If we apply Theorem 2 to Lebesgue measure on the real line, we see that every standard real number is in the null set \( A_0 \).

Let \( \Phi_1 = \{ \mu \in \Phi : \mu(X) = 1 \text{ and } \forall B \subseteq \mathcal{M}, \mu(B) = 0 \text{ or } \mu(B) = 1 \} \). For each \( j \in I \), let \( \delta_j \in E \) be defined by setting \( \delta_j = 0 \) if \( i \neq j \) and \( \delta_j = 1 \).

**Theorem 3.** For each \( j \in I \), \( \varphi(\delta_j) \in \Phi_1 \), and for each \( \mu \in \Phi_1 \), \( U(\mu) = \delta_j \) for some \( j \in I \). Moreover, if \( \{ x \} \subseteq \mathcal{M} \) for each standard point \( x \in X \), then the following are equivalent statements:
(i) Given \( j \in I \), \( \varphi(\delta_j) \in \Phi_1 \) iff \( A_j \neq \{ x \} \) for any standard point \( x \in X \).
(ii) Every free \( \mathcal{M} \)-measurable ultrafilter \( \mathcal{F} \subseteq \mathcal{M} \) contains a chain \( B_1 \supseteq B_2 \supseteq \cdots \), with \( \bigcap_{n=1}^{\infty} B_n = \emptyset \).

If \( \mu \) is a nonnegative finitely additive measure on \( (X, \mathcal{M}) \) and \( f \geq 0 \) is \( \mu \)-integrable on \( X \), then for each \( B \subseteq \mathcal{M} \),
\[
\int_B f \, d\mu \simeq \sum_{i \in I_B} \left( \inf_{x \in A_i} *f(x) \right) *\mu(A_i).
\]

We can relate integration on \( X \) to the inner product \( \cdot \cdot \) in \( E \) as follows:

**Theorem 4.** If \( f \in MB \) and \( \mu \in \Phi \), then for each \( B \subseteq \mathcal{M} \),
\[
\int_B f \, d\mu = \sum_{i \in I_B} *f(c_P(i)) *\mu(A_i).
\]

In particular, \( \int_X f \, d\mu \simeq T(f) \cdot U(\mu) \).

In general, Theorem 4 is false for unbounded functions \( f \in M \). One can, however, find for each \( f \in M \) an \( \omega \in \mathcal{M} \) such that \( *f_\omega = -\omega \lor f \land \omega \), then for each \( i \in I \), \( \sup_{x \in A_i} *f_\omega(x) - \inf_{x \in A_i} *f_\omega(x) \simeq 0 \). If \( \mu \in \Phi \) and \( f \) is \( \mu \)-integrable, then
\[
\int_X f \, d\mu \simeq \sum_{i \in B} *f_\omega(c_P(i)) *\mu(A_i).
\]

3. The space \( L_\infty \) and its conjugate space. Let \( \mathcal{F} \) be a proper subfamily of \( \mathcal{M} \) such that \( \mathcal{F} \) is closed under the formation of countable
unions and every \( \mathcal{M} \)-measurable subset of an element of \( \mathcal{X} \) is an element of \( \mathcal{X} \). For each \( f \in M \), set

\[
\|f\|_\infty = \inf \{ \alpha \in \mathbb{R} : \{ x \in X : |f(x)| > \alpha \} \in \mathcal{X} \},
\]

and let \( M_0 = \{ f \in M : \|f\|_\infty < +\infty \} \). We say that two functions \( f \) and \( g \) in \( M_0 \) are equivalent if \( \|f - g\|_\infty = 0 \), and we let \( L_\infty \) denote the usual Banach space of equivalence classes in \( M_0 \) with norm \( \| \cdot \|_\infty \).

Given \( \mathcal{X} \), let \( I_0 = \{ i \in I : A_i \in \mathcal{X} \} \). Clearly, if \( B \in \mathcal{X} \), \( I_B \subset I_0 \). For each \( f \in M_0 \), let \( T_0(f) \) be that element of \( E \) such that \( T_0(f)(i) = *f(c_p(i)) \) for \( i \in I - I_0 \) and \( T_0(f)(i) = 0 \) for \( i \in I_0 \). Given \( f \) and \( g \) in \( M_0 \), \( T_0(f) \cong T_0(g) \Rightarrow \|f - g\|_\infty = 0 \Rightarrow T_0(f) = T_0(g) \). Moreover, \( \|f\|_\infty \approx \max_{i \in I} |T_0(f)(i)| \). We may, therefore, consider \( T_0 \) to be a mapping of \( L_\infty \) into \( E \); this mapping preserves addition and multiplication by standard real numbers.

For each functional \( F \) in the dual space \( L_\infty^* \) of \( L_\infty \), let \( V(F) \) be the element of \( E \) such that for all \( i \in I \), \( V(F)(i) = *F(x_{A_i}) \), and let \( \mu_F = \phi(V(F)) \). It is easy to see that \( U(\mu_F) = V(F) \). Yosida and Hewitt's representation of \( L_\infty^* \) ([6, p. 53]) now has the following form:

**Theorem 5.** Let \( \Phi_0 \) be the normed vector space \( \{ \mu \in \Phi_0 : \mu(B) = 0, \forall B \in \mathcal{X} \} \) with norm given by \( \| \mu \| = |\mu|(X) \). For each \( F \in L_\infty^* \), let \( \Theta(F) = \mu_F \). Then \( \Theta \) is an isometric isomorphism from the Banach space \( L_\infty^* \) onto \( \Phi_0 \), and for each \( F \in L_\infty^* \) and \( f \in L_\infty \) we have

\[
F(f) = \int_X f \, d\mu_F \simeq V(F) \cdot T_0(f).
\]

**Corollary.** A nonzero functional \( F \in L_\infty^* \) is multiplicative iff \( U(\mu_F) = \delta^j \) for some \( j \in I - I_0 \).

Assume now that there is a nonnegative \( \mu \in \Phi_0 \) such that \( \mathcal{X} = \{ B \in \mathcal{M} : \mu(B) = 0 \} \). If \( f \in L_\infty \) and \( \nu \in \Phi_0 \) has the value \( \nu(B) = \int_B f \, d\mu \) for each \( B \in \mathcal{M} \), then for each \( i \in I - I_0 \), \( *f(c_p(i)) \simeq *\nu(A_i)/\mu(A_i) \). To apply this result to probability theory, assume that \( \mu(X) = 1 \) and choose a \( \sigma \)-algebra \( \mathcal{M}_0 \subset \mathcal{M} \). There is a \( * \)-finite, \( *\mathcal{M}_0 \)-measurable partition \( P_1 \) of \( *X \) such that \( P_1 \) is finer than any standard, finite \( \mathcal{M}_0 \)-measurable partition of \( X \) and such that for each \( C \in P_1 \), \( C = \bigcup \{ A_i \in P : A_i \subset C \} \). If \( Y \in MB \) and \( E(Y, \mathcal{M}_0) \) is the conditional expectation of \( Y \) with respect to \( \mathcal{M}_0 \), then for each \( C \in P_1 \) with \( \mu(C) \neq 0 \) and for each \( x \in C \),

\[
*E(Y, \mathcal{M}_0)(x) \simeq \left[ \sum_{A_i \in P : A_i \subset C} *Y(c_p(i)) *\mu(A_i) \right]/*\mu(C).
\]
REFERENCES


University of Illinois, Urbana, Illinois 61801