A NONSTANDARD REPRESENTATION OF MEASURABLE SPACES AND $L_\omega$

BY PETER A. LOEB

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The results given in this note were obtained by applying to measure theory the methods of nonstandard analysis developed by Abraham Robinson [5]. Amplifications of these results with proofs will be published elsewhere. It is shown here that there are linear mappings from an arbitrary, real $L_\omega$ space and its dual $L_\omega^*$ into Euclidean $\omega$-space $E^\omega$, where $\omega$ is an infinite integer. Finite valued, finitely additive measures on the underlying measurable space are also mapped onto elements of $E^\omega$, and integrals are infinitesimally close to the corresponding inner products in $E^\omega$. Yosida and Hewitt's representation of $L_\omega^*$ [6] is an immediate consequence of these results.

In general, we use Robinson's notation [5]. If we have an enlargement of a structure that contains the set $R$ of real numbers, then $*R$ denotes the set of nonstandard real numbers and $*N$, the set of nonstandard natural numbers. A set $S$ is called *finite if there is an internal bijection from an initial segment of $*N$ onto $S$; a *finite set has all of the "formal" properties of a finite set. Given $b$ and $c$ in $*R$, we write $b \sim c$ if $b - c$ is in the monad of 0; when $b$ is finite, we write $\circ b$ for the unique, standard real number in the monad of $b$.

1. The partition $P$ and bounded measurable functions. Let $X$ be an infinite set and $\mathcal{M}$ an infinite $\sigma$-algebra of subsets of $X$. Fix an enlargement of a structure that contains $X$, $\mathcal{M}$, and the extended real numbers. There is a *finite, $*\mathcal{M}$-measurable partition $P$ of $*X$ such that $P$ is finer than any finite $\mathcal{M}$-measurable partition of $X$. That is, $P \subset *\mathcal{M}$ has the following properties:

(i) There is an infinite integer $\omega_P \in *N$ and an internal bijection from $I = \{i \in *N : 1 \leq i \leq \omega_P\}$ onto $P$. Thus we may write $P = \{A_i : i \in I\}$.

(ii) If $i$ and $j$ are in $I$ and $i \neq j$, then $A_i \neq \emptyset$ and $A_i \cap A_j = \emptyset$.


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(iii) \( *X = \bigcup_{i \in I} A_i \).

(iv) For each \( B \subseteq \mathcal{M} \), let \( I_B = \{ i \in I : A_i \subseteq *B \} \). Then \( I_B \) is *finite, and \( *B = \bigcup_{i \in I_B} A_i \).

(v) Let \( M \) be the set of \( \mathcal{M} \)-measurable functions on \( X \), and \( MB \), the set of bounded functions in \( M \). For each \( f \in MB \) and \( i \in I \), \( \sup_{x \in A_i} f(x) - \inf_{x \in A_i} f(x) \leq 0 \).

Given the partition \( P \), we let \( E \) denote the set of all internal mappings from \( I \) into \( *R \). The set \( E \) has all of the "formal" properties of Euclidean \( n \)-space. We shall write \( x_i \) instead of \( x(i) \) for \( x \in E \) and \( i \in I \), and we shall write \( x \equiv y \) if \( x, y \in E \) and \( x_i \equiv y_i \), \( \forall i \in I \). Let \( c_P \) denote a fixed internal choice function defined on \( I \) with \( c_P(i) \in A_i \in P \) for each \( i \in I \). Let \( T \) denote the mapping from \( MB \) into \( E \) defined by setting \( T(f)(i) = *f(c_P(i)) \) for each \( f \in MB \) and \( i \in I \).

**Proposition 1.** Given \( f, g \) in \( MB \) and \( \alpha, \beta \) in \( R \), \( T(\alpha f + \beta g) = \alpha T(f) + \beta T(g) \) and \( T(f) \neq T(g) \) if \( f \neq g \).

2. **Measures and integration.** Let \( \Phi(X, \mathcal{M}) \), or simply \( \Phi \), denote the set of all finitely additive real-valued functions \( \mu \) on \( \mathcal{M} \) such that \( \sup_{B \subseteq \mathcal{M}} |\mu(B)| < +\infty \). Let \( U \) be the mapping of \( \Phi \) into \( E \) defined by setting \( U(\mu)(i) = *\mu(A_i) \) for each \( \mu \in \Phi \) and \( i \in I \). Clearly, \( U \) preserves addition and multiplication by real numbers. Conversely, if \( e \in E \) and both \( \sum_{i \in I} (e_i \vee 0) \) and \( \sum_{i \in I} (-e_i \vee 0) \) are finite in \( *R \), let \( \varphi(e) \) be that element of \( \Phi \) such that for each \( B \subseteq \mathcal{M} \), \( \varphi(e)(B) = \circ \sum_{i \in I_B} e_i \).

(Note that we are writing \( \sum \) instead of \( * \sum \) for the extension of the summation operator.) For each \( \mu \in \Phi \), \( U(\mu) = \mu \), but in general, \( U(\varphi(e)) \neq e \). If \( \mu \) and \( \nu \) are in \( \Phi \), then \( U(\mu) \wedge U(\nu) \equiv U(\mu \wedge \nu) \), and \( \circ \sum_{i \in I} U(\mu)(i) = |\mu| \circ (X) \).

Let \( \Phi_e \) and \( \Phi_p \) be, respectively, the set of countably additive and the set of purely finitely additive elements of \( \Phi \). Yosida and Hewitt’s Theorem 1.19 [6] has the following extension:

**Theorem 1.** There is a set \( K \in \mathcal{M} \) such that for all \( \mu \in \Phi_e \), \( |\mu| (K) \approx 0 \) and for all \( \nu \in \Phi_p \), \( |\nu| (X - K) = 0 \).

Without loss of generality, we assume that \( K = U \{ A_i \subseteq P : A_i \subseteq K \} \).

If \( \mu = \mu_e + \mu_p \) is the decomposition of an element \( \mu \) in \( \Phi = \Phi_e \oplus \Phi_p \), then when \( A_i \subseteq K \), \( U(\mu)(i) = U(\mu_e)(i) \) and when \( A_i \subseteq X - K \), \( U(\mu)(i) \approx U(\mu_p)(i) \). We next show that there is a “maximum” null set for each \( \mu \in \Phi^+ \), and we extend the Hahn decomposition theorem for countably additive signed measures.

**Theorem 2.** Let \( \mu \) be an arbitrary, finitely additive signed measure on \( (X, \mathcal{M}) \). Let
\[ A_+ = \bigcup \{ A_i \in P : \mu(A_i) > 0 \}, \quad A_- = \bigcup \{ A_i \in P : \mu(A_i) < 0 \}, \]

and

\[ A_0 = \bigcup \{ A_i \in P : \mu(A_i) = 0 \}. \]

Then \( \mu(A_0) = 0 \), and for each \( \mu \)-null set \( B \subseteq \mathcal{M} \), \( B \subseteq A_0 \). If there exists a \( \mu \)-positive set \( B_+ \) and a \( \mu \)-negative set \( B_- \) in \( \mathcal{M} \) with \( X = B_+ \cup B_- \) and \( B_+ \cap B_- = \emptyset \), then \( A_+ \subseteq *B_+ \), \( A_- \subseteq *B_- \), and each \( A_i \in P \) is either a \( *\mu \)-positive set or a \( *\mu \)-negative set.

If we apply Theorem 2 to Lebesgue measure on the real line, we see that every standard real number is in the null set \( A_0 \).

Let \( \Phi_1 = \{ \mu \in \Phi : \mu(X) = 1 \text{ and } \forall B \in \mathcal{M}, \mu(B) = 0 \text{ or } \mu(B) = 1 \} \). For each \( j \in I \), let \( \delta^j \in E \) be defined by setting \( \delta^j_i = 0 \) if \( i \neq j \) and \( \delta^j_i = 1 \).

**Theorem 3.** For each \( j \in I \), \( \varphi(\delta^j) \in \Phi_1 \), and for each \( \mu \in \Phi_1 \), \( U(\mu) = \delta^j \) for some \( j \in I \). Moreover, if \( \{ x \} \in \mathcal{M} \) for each standard point \( x \in X \), then the following are equivalent statements:

(i) Given \( j \in I \), \( \varphi(\delta^j) \in \Phi_1 \), and \( A_j \neq \{ x \} \) for any standard point \( x \in X \). 

(ii) Every free \( \mathcal{M} \)-measurable ultrafilter \( \mathcal{F} \subseteq \mathcal{M} \) contains a chain \( B_1 \supseteq B_2 \supseteq \cdots \), with \( \bigcap_{n=1}^{\infty} B_n = \emptyset \).

If \( \mu \) is a nonnegative finitely additive measure on \( (X, \mathcal{M}) \) and \( f \geq 0 \) is \( \mu \)-integrable on \( X \), then for each \( B \in \mathcal{M} \),

\[ \int_B f \, d\mu = \sum_{i \in B} \left( \inf_{x \in A_i} *f(x) \right) *\mu(A_i). \]

We can relate integration on \( X \) to the inner product 

\( \cdot \nabla \nabla \) in \( E \) as follows:

**Theorem 4.** If \( f \in MB \) and \( \mu \in \Phi \), then for each \( B \in \mathcal{M} \),

\[ \int_B f \, d\mu = \sum_{i \in B} *f(c_P(i)) *\mu(A_i). \]

In particular, \( \int_X f \, d\mu \simeq T(f) \cdot U(\mu) \).

In general, Theorem 4 is false for unbounded functions \( f \in M \). One can, however, find for each \( f \in M \) an \( \omega \in \text{ran} F \) such that if \( *f_{\omega} = -\omega \nabla *f \nabla \omega \), then for each \( i \in I \), \( \sup_{x \in A_i} *f_{\omega}(x) - \inf_{x \in A_i} *f_{\omega}(x) \approx 0 \). If \( \mu \in \Phi \) and \( f \) is \( \mu \)-integrable, then

\[ \int_X f \, d\mu \approx \sum_{i \in B} *f_{\omega}(c_P(i)) *\mu(A_i). \]

3. **The space \( L_\infty \) and its conjugate space.** Let \( \mathcal{F} \) be a proper subfamily of \( \mathcal{M} \) such that \( \mathcal{F} \) is closed under the formation of countable...
unions and every $\mathcal{M}$-measurable subset of an element of $\mathcal{N}$ is an element of $\mathcal{N}$. For each $f \in M$, set

$$
\|f\|_\infty = \inf\{\alpha \in \mathbb{R} : \{x \in X : |f(x)| > \alpha\} \in \mathcal{N}\},
$$

and let $M_0 = \{f \in M : \|f\|_\infty < +\infty\}$. We say that two functions $f$ and $g$ in $M_0$ are equivalent if $\|f - g\|_\infty = 0$, and we let $L_\infty$ denote the usual Banach space of equivalence classes in $M_0$ with norm $\|\cdot\|_\infty$.

Given $\mathcal{N}$, let $I_0 = \{i \in I : A_i \in \mathcal{N}\}$. Clearly, if $B \in \mathcal{N}$, $I_B \subset I_0$. For each $f \in M_0$, let $T_0(f)$ be that element of $E$ such that $T_0(f)(i) = f(c_P(i))$ for $i \in I - I_0$ and $T_0(f)(i) = 0$ for $i \in I_0$. Given $f$ and $g$ in $M_0$, $T_0(f) \cong T_0(g) \Rightarrow \|f - g\|_\infty = 0 \Rightarrow T_0(f) = T_0(g)$. Moreover, $\|f\|_\infty \cong \max_{i \in I} |T_0(f)(i)|$. We may, therefore, consider $T_0$ to be a mapping of $L_\infty$ into $E$; this mapping preserves addition and multiplication by standard real numbers.

For each functional $F$ in the dual space $L_\infty^*$ of $L_\infty$, let $V(F)$ be the element of $E$ such that for all $i \in I$, $V(F)(i) = *F(\chi_{A_i})$, and let $\mu_F = \phi(V(F))$. It is easy to see that $U(\mu_F) = V(F)$. Yosida and Hewitt's representation of $L_\infty^*$ ([6, p. 53]) now has the following form:

**Theorem 5.** Let $\Phi_0$ be the normed vector space $\{\mu \in \Phi : \mu(B) = 0, \forall B \in \mathcal{N}\}$ with norm given by $\|\mu\| = |\mu|(X)$. For each $F \in L_\infty^*$, let $\Theta(F) = \mu_F$. Then $\Theta$ is an isometric isomorphism from the Banach space $L_\infty^*$ onto $\Phi_0$, and for each $F \in L_\infty^*$ and $f \in L_\infty$ we have

$$
F(f) = \int_X f \, d\mu_F \simeq V(F) \cdot T_0(f).
$$

**Corollary.** A nonzero functional $F \in L_\infty^*$ is multiplicative iff $U(\mu_F) = \delta_j$ for some $j \in I - I_0$.

Assume now that there is a nonnegative $\mu \in \Phi_0$ such that $\mathcal{N} = \{B \in \mathcal{N} : \mu(B) = 0\}$. If $f \in L_\infty$ and $\nu \in \Phi_0$ has the value $\nu(B) = \int_B f \, d\mu$ for each $B \in \mathcal{N}$, then for each $i \in I - I_0$, $*f(c_P(i)) \simeq *\nu(\chi_{A_i}) / *\mu(\chi_{A_i})$. To apply this result to probability theory, assume that $\mu(X) = 1$ and choose a $\sigma$-algebra $\mathcal{M}_i \subset \mathcal{N}$. There is a $*\sigma$-finite, $*\mathcal{M}_1$-measurable partition $P_1$ of $*X$ such that $P_1$ is finer than any standard, finite $\mathcal{M}_1$-measurable partition of $X$ and such that for each $C \in P_1$, $C = \bigcup \{A_i \in P : A_i \subset C\}$. If $Y \in MB$ and $E(Y, \mathcal{M}_i)$ is the conditional expectation of $Y$ with respect to $\mathcal{M}_i$, then for each $C \in P_1$ with $\mu(C) \neq 0$ and for each $x \in C$,

$$
*E(Y, \mathcal{M}_i)(x) \simeq \left[ \sum_{A_i \in P_1 : A_i \subset C} *Y(c_P(i)) *\mu(A_i) \right] / *\mu(C).
$$
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UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801