K-THEORY OF A SPACE WITH COEFFICIENTS IN A (DISCRETE) RING

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In [2], [3], S. Gersten has introduced higher K-groups of a ring which satisfy properties analogous to those of a generalized homology theory in a suitably defined homotopy category of rings [1]. In this announcement we use Gersten’s K-groups to define for a ring R a generalized cohomology theory $K^*_R( )$, analogous to the Atiyah-Hirzebruch K-theory, on the category of finite simplicial sets so that $K^*_R(pt) = K^*_R$, where $K^*_R$ are Gersten’s stable K-groups of the ring R. If R is suitably restricted, in particular if it is commutative and regular, the theory $K^*_R( )$ will have products and Adams operations. One may also define, using the continuous theory in [6], a K-theory $K^*_A( )$ with coefficients in a Banach ring A. This theory coincides with the Atiyah-Hirzebruch theory for $\Lambda = R, C, or H$. We give here an outline of proofs. A full account will appear elsewhere.

1. Definition of the theory. We recall the definition of Gersten’s theory as given in [5]. Let R be a ring (without unit). The functor $R \rightarrow R[t]$ together with the natural transformations $R[t] \rightarrow R$ via “$t \rightarrow 1$”, and $R[t] \rightarrow R[t, t']$ via $t \rightarrow tt'$ define a cotriple in the category of rings. If $ER$ is the ideal $R[t]/R$, then the restriction of those maps makes the functor $R \rightarrow ER$ a cotriple. Associated to these cotriples are canonical simplicial rings $R[T]$ and $ER$ with

$$R[T]_n = R[t_0, \ldots, t_n], \quad ER_n = E^{n+1}R.$$ 

Let $QR$ be the simplicial ring

$$QR = R[T]/ER.$$ 

One has

$$K^{i-1}R = \pi_i GL QR$$

where GL denotes the general linear group functor. This K-theory of rings is stabilized as follows [3]. Let $IR$ be the kernel of $R[t, t^{-1}] \rightarrow R$. Then there is a natural homomorphism


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analogous to the Bott map, which is an isomorphism when \( R \) is \( K \)-regular.

Put

\[
K^i R = \lim_{\longleftarrow} K^{i-n} \Gamma^n R, \quad -\infty < i < \infty.
\]

Then if \( R \to S \to T \) is a Gl-fibration \([2]\), there is a long exact sequence

\[
\cdots \to K^i T \to K^i S \to K^i R \to K^{i+1} R \to \cdots.
\]

Now to define a cohomology theory for simplicial sets, we will give a contravariant functor \((\cdot ; R)\) from simplicial sets to rings which

1. sends coproducts to products,
2. sends cofibrations to Gl-fibrations,
3. sends a point to \( R \).

For \( X \) and \( Y \) simplicial sets, let \( A(X;Y) \) denote the set of all simplicial maps from \( X \) to \( Y \). Put

\[
(X;R) = A(X;QR).
\]

Then \( (X;R) \) is a ring and \( (pt;R) = R \), since \( QR_0 = R \).

**Definition 1.1.** For \( X \) a finite simplicial set, \( K^*_R(X) = K^*_R(X;R) \).

The long exact sequence of a cofibration arises from

**Proposition 1.2.** If \( Y \to X \to X/Y \) is a cofibration of simplicial sets then

\[
(X/Y;R) \to (X;R) \to (Y;R)
\]

is a Gl-fibration.

This proposition follows from the following properties of the functor \( \Delta(\cdot;\cdot) \).

**Lemma 1.3.** If \( F \) is a functor which is left exact and preserves products then

\[
\Delta(X;FY) = F\Delta(X;Y).
\]

**Proof.** Follows from the fact that \( F \) preserves equalizers.

**Lemma 1.4.** If \( Y \) is a simplicial object in a category with a forgetful functor to sets and \( Y \) is contractible as a set complex then \( \Delta(\cdot;Y) \) is an exact functor.
To verify the homotopy axiom for the theory we must prove

**Proposition 1.5.** If \( X \to Y \) is a map of finite simplicial sets which induces an isomorphism \( H_\ast(X; \mathbb{Z}) \to H_\ast(Y; \mathbb{Z}) \), then \( j^! : K_R^\ast(Y) \to K_R^\ast(X) \) is an isomorphism.

This proposition follows immediately from the fact that we have an analogue of the Atiyah-Hirzebruch spectral sequence defined intrinsically in the theory \( K_R^\ast \) as follows.

Let \( X^n \) be the \( n \)-skeleton of \( X \). We have a tower of \( \text{GL} \)-fibrations

\[
\cdots \to (X^n; R) \to (X^{n-1}; R) \to \cdots \to (X^0; R).
\]

The long exact \( K_\ast^R \)-theory sequences of these fibrations define a homology exact couple. The spectral sequence of that couple converges strongly to \( K_R^\ast(X) \). One has \( E^p_1 = K_\ast^{p+q}(X^p/X^{p-1}; R) \). By a brute force calculation

**Lemma 1.6.** \( K_\ast^{p+q}(X^p/X^{p-1}; R) = \bigoplus \sigma \cdot K_\ast^{p+q}R \), where the sum runs over all nondegenerate \( p \)-simplexes \( \sigma \) of \( X \).

Standard diagram chases now establish

**Theorem 1.7.** There is a natural spectral sequence \( \{ E_r \} \) converging to \( K_R^\ast(X) \) with

\[
E_2^{p,q} = H^p(X; K_\ast^q R).
\]

Thus

**Theorem 1.8.** \( K_R^\ast( ) \) is a generalized cohomology theory on the category of finite simplicial sets.

In addition,

**Theorem 1.9.** \( K_R^\ast( ) \) depends only on the ring homotopy type of \( R \) and if \( R \to S \to T \) is a \( \text{GL} \)-fibration of rings there is a natural exact triangle of theories

\[
K_R^\ast( ) \longrightarrow K_S^\ast( ) \quad \delta \quad K_T^\ast( )
\]

where \( \delta \) has degree \(+1\).

2. **Products and Adams operations.** Let \( R \) and \( T \) be rings, \( X \) and \( Y \) simplicial sets. We then have a pairing

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\((X; R) \otimes z (Y; T) \xrightarrow{\phi} (X \times Y; R \otimes T)\)
given by 
\[\phi(\alpha \otimes \beta)(x, y) = \alpha(x) \otimes \beta(y).\]

Using the product structure in \(\kappa^*_X( )\) one has

**Theorem 2.1.** There is a natural graded associative pairing

\[\kappa^*_R(X) \otimes \kappa^*_T(Y) \rightarrow \kappa^*_{R \otimes T}(X \times Y).\]

If \(R\) is a commutative ring there is a natural graded commutative ring structure on \(\kappa^*_R(X)\) arising from the diagonal \(\Delta: X \rightarrow X \times X\).

Now suppose that \(R\) is a \(K\)-regular ring \([2]\). From a truncated version of the spectral sequence of Theorem 1.7 one has

**Theorem 2.2.** If \(R\) is \(K\)-regular,

\[K^i(X; R) = K^i(X; R)\]

for \(i \leq 0\).

Now the theory \(K^i\) has Adams operations which are graded ring homomorphisms. Let \(\kappa^-_R( )\) be the nonpositive graded part of \(\kappa^*_R( )\). Then

**Theorem 2.3.** If \(R\) is \(K\)-regular there are natural graded ring morphisms

\[\psi^k: \kappa^-_R(X) \rightarrow \kappa^-_R(X)\]

for \(k \geq 0\). The \(\psi^k\) commute with the boundary of the long exact sequence of a cofibration when that makes sense.

**Remark 2.4.** Using the continuous polynomials of \([6]\) one may define a theory \(\kappa^*_\Lambda( )\) for \(\Lambda\) a valuation ring. For \(\Lambda = R, C\) or \(H\) there is a natural equivalence

\[\kappa^*_\Lambda( ) \rightarrow \kappa^*_\Lambda( )\]

where \(\kappa^*_\Lambda\) is the \(K\)-theory of Atiyah and Hirzebruch. It would be interesting to know the coefficient group \(\kappa^*_\Lambda(pt) = K^*\Lambda\) for \(\Lambda = Q_p\) or \(Z_p\).

**Remark 2.5.** The ring complex \(QR\) above may be replaced by the nicer ring complex \(\Delta R\) where

\[\Delta R_n = R[t_0, \ldots, t_n]/t_0 + \cdots + t_n - 1,\]
and
\[
\begin{align*}
    d_i d_j &= t_i, & i > j, \\
    &= 0, & i = j, \\
    &= t_{j-1}, & i < j, \\
    s_i d_j &= t_i, & i > j, \\
    &= t_i + t_{j+1}, & i = j, \\
    &= t_{j+1}, & i < j.
\end{align*}
\]

One may now redefine \((X; R)\) as the ring of simplicial maps of \(X\) to \(\Delta R\). The same \(K\)-theory for \(X\) now arises in view of

**Proposition 2.6.** \(\pi_i \text{ Gl } \Delta R = K^{-i-1}R, i \geq 0\).

This proposition is proved by showing that \(\pi_i \text{ Gl } \Delta R\) satisfies the axioms for \(K^{-i-1}R [2]\).

**References**


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