K-THEORY OF A SPACE WITH COEFFICIENTS IN A (DISCRETE) RING

BY DAVID L. RECTOR

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In [2], [3], S. Gersten has introduced higher $K$-groups of a ring which satisfy properties analogous to those of a generalized homology theory in a suitably defined homotopy category of rings [1]. In this announcement we use Gersten’s $K$-groups to define for a ring $R$ a generalized cohomology theory $K^*_R(\cdot)$, analogous to the Atiyah-Hirzebruch $K$-theory, on the category of finite simplicial sets so that $K^*_R(pt) = K^*_R$, where $K^*_R$ are Gersten’s stable $K$-groups of the ring $R$. If $R$ is suitably restricted, in particular if it is commutative and regular, the theory $K^*_R(\cdot)$ will have products and Adams operations. One may also define, using the continuous theory in [6], a $K$-theory $K^*_\Lambda(\cdot)$ with coefficients in a Banach ring $\Lambda$. This theory coincides with the Atiyah-Hirzebruch theory for $\Lambda = R$, $C$, or $H$. We give here an outline of proofs. A full account will appear elsewhere.

1. Definition of the theory. We recall the definition of Gersten’s theory as given in [5]. Let $R$ be a ring (without unit). The functor $R \rightarrow R[t]$ together with the natural transformations $R[t] \rightarrow R$ via “$t \rightarrow 1$”, and $R[t] \rightarrow R[t, t']$ via $t \rightarrow tt'$ define a cotriple in the category of rings. If $ER$ is the ideal $R[t, t']$, then the restriction of those maps makes the functor $R \rightarrow ER$ a cotriple. Associated to these cotriples are canonical simplicial rings $R[T]$ and $ER$ with

$$R[T]_n = R[t_0, \ldots, t_n], \quad ER_n = E^{n+1}R.$$ 

Let $QR$ be the simplicial ring

$$QR = R[T]/ER.$$

One has

$$K^{i-1}R = \pi_i \text{ Gl } QR$$

where Gl denotes the general linear group functor. This $K$-theory of rings is stabilized as follows [3]. Let $IR$ be the kernel of $R[t, t^{-1}] \rightarrow R$. Then there is a natural homomorphism

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analogous to the Bott map, which is an isomorphism when \( R \) is \( K \)-regular.

Put

\[
K^i \beta R = \text{inj lim}_n K^{i-n} \Gamma^n R, \quad -\infty < i < \infty.
\]

Then if \( R \rightarrow S \rightarrow T \) is a Gl-fibration \([2]\), there is a long exact sequence

\[
\cdots \rightarrow K^i T \rightarrow K^i S \rightarrow K^i R \rightarrow K^{i+1} \rightarrow \cdots.
\]

Now to define a cohomology theory for simplicial sets, we will give a contravariant functor \((\ ; \ R)\) from simplicial sets to rings which

1. sends coproducts to products,
2. sends cofibrations to Gl-fibrations,
3. sends a point to \( R \).

For \( X \) and \( Y \) simplicial sets, let \( \Delta(X; Y) \) denote the set of all simplicial maps from \( X \) to \( Y \). Put

\[
(X; R) = \Delta(X; QR).
\]

Then \( (X; R) \) is a ring and \( (pt; R) = R \), since \( QR_0 = R \).

**Definition 1.1.** For \( X \) a finite simplicial set, \( K^*_R(X) = K^*_R(X; R) \).

The long exact sequence of a cofibration arises from

**Proposition 1.2.** If \( Y \rightarrow X \rightarrow X/Y \) is a cofibration of simplicial sets then

\[
(X/Y; R) \rightarrow (X; R) \rightarrow (Y; R)
\]

is a Gl-fibration.

This proposition follows from the following properties of the functor \( \Delta(\ ; \ ) \).

**Lemma 1.3.** If \( F \) is a functor which is left exact and preserves products then

\[
\Delta(X; FY) = F \Delta(X; Y).
\]

**Proof.** Follows from the fact that \( F \) preserves equalizers.

**Lemma 1.4.** If \( Y \) is a simplicial object in a category with a forgetful functor to sets and \( Y \) is contractible as a set complex then \( \Delta(\ Y) \) is an exact functor.
To verify the homotopy axiom for the theory we must prove

**Proposition 1.5.** If \( X \to Y \) is a map of finite simplicial sets which induces an isomorphism \( H_\bullet(X; \mathbb{Z}) \to H_\bullet(Y; \mathbb{Z}) \), then \( f^! : K^*_R(Y) \to K^*_R(X) \) is an isomorphism.

This proposition follows immediately from the fact that we have an analogue of the Atiyah-Hirzebruch spectral sequence defined intrinsically in the theory \( K^*_R \) as follows.

Let \( X^n \) be the \( n \)-skeleton of \( X \). We have a tower of Gl-fibrations

\[
\cdots \to (X^n; R) \to (X^{n-1}; R) \to \cdots \to (X^0; R).
\]

The long exact \( K^*_R \)-theory sequences of these fibrations define a homology exact couple. The spectral sequence of that couple converges strongly to \( K^*_R(X) \). One has \( E_1^{p,q} = K^{p+q}_R(X^p/X^{p-1}; R) \). By a brute force calculation

**Lemma 1.6.** \( K^{p+q}_R(X^p/X^{p-1}; R) = \bigoplus \sigma K^{p+q}_R \), where the sum runs over all nondegenerate \( p \)-simplexes \( \sigma \) of \( X \).

Standard diagram chases now establish

**Theorem 1.7.** There is a natural spectral sequence \( \{ E_r \} \) converging to \( K^*_R(X) \) with

\[
E_2^{p,q} = H^p(X; K^q_R R).
\]

Thus

**Theorem 1.8.** \( K^*_R(\ ) \) is a generalized cohomology theory on the category of finite simplicial sets.

In addition,

**Theorem 1.9.** \( K^*_R(\ ) \) depends only on the ring homotopy type of \( R \) and if \( R \to S \to T \) is a Gl-fibration of rings there is a natural exact triangle of theories

\[
K^*_R(\ ) \to K^*_S(\ ) \to K^*_T(\ ),
\]

where \( \delta \) has degree \( +1 \).

2. **Products and Adams operations.** Let \( R \) and \( T \) be rings, \( X \) and \( Y \) simplicial sets. We then have a pairing
Given by

\[ \phi(\alpha \otimes \beta)(x, y) = \alpha(x) \otimes \beta(y). \]

Using the product structure in \( K_\mathbb{Z}(\ ) \) [4] one has

**Theorem 2.1.** There is a natural graded associative pairing

\[ K_R^*(X) \otimes K_T^*(Y) \rightarrow K_{R \otimes T}^*(X \times Y). \]

If \( R \) is a commutative ring there is a natural graded commutative ring structure on \( K_R^*(X) \) arising from the diagonal \( \Delta: X \rightarrow X \times X \).

Now suppose that \( R \) is a \( K \)-regular ring [2]. From a truncated version of the spectral sequence of Theorem 1.7 one has

**Theorem 2.2.** If \( R \) is \( K \)-regular,

\[ K^i_\mathbb{Z}(X; R) = K^i(X; R) \]

for \( i \leq 0 \).

Now the theory \( K^i \) has Adams operations which are graded ring homomorphisms. Let \( K_R^*(\ ) \) be the nonpositive graded part of \( K_R^*(\ ) \). Then

**Theorem 2.3.** If \( R \) is \( K \)-regular there are natural graded ring morphisms

\[ \psi^k : K_R^-(X) \rightarrow K_R^-(X) \]

for \( k \geq 0 \). The \( \psi^k \) commute with the boundary of the long exact sequence of a cofibration when that makes sense.

**Remark 2.4.** Using the continuous polynomials of [6] one may define a theory \( K^*_\Lambda(\ ) \) for \( \Lambda \) a valuation ring. For \( \Lambda = \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) there is a natural equivalence

\[ K^*_\Lambda(\ ) \rightarrow K^*_\Lambda(\ ) \]

where \( K^*_\Lambda \) is the \( K \)-theory of Atiyah and Hirzebruch. It would be interesting to know the coefficient group \( K^*_\Lambda(pt) = K^*_\Lambda \) for \( \Lambda = \mathbb{Q}_p \) or \( \mathbb{Z}_p \).

**Remark 2.5.** The ring complex \( QR \) above may be replaced by the nicer ring complex \( DR \) where

\[ \Delta R_n = R[t_0, \ldots, t_n]/t_0 + \cdots + t_n - 1, \]
and

\[
\begin{align*}
    d_i d_j & = t_j, & i > j, \\
    & = 0, & i = j, \\
    & = t_{j-1}, & i < j, \\
    s_i d_i & = t_i, & i > j, \\
    & = t_i + t_{i+1}, & i = j, \\
    & = t_{i+1}, & i < j.
\end{align*}
\]

One may now redefine \((X; R)\) as the ring of simplicial maps of \(X\) to \(\Delta R\). The same \(K\)-theory for \(X\) now arises in view of

**Proposition 2.6.** \(\pi_i \text{ GL} \Delta R = K^{-i-1}R, i \geq 0\).

This proposition is proved by showing that \(\pi_i \text{ GL} \Delta R\) satisfies the axioms for \(K^{-i-1}R\) [2].

**References**


**Rice University, Houston, Texas 77001**