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CHARACTERIZATIONS OF BOUNDED MEAN OSCILLATION

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BMO (bounded mean oscillation) is the Banach space of all functions $f \in L^1_{\mathrm{loc}}(R^n)$ for which

$$\|f\|_{\mathrm{BMO}} = \sup_Q \left(\frac{1}{|Q|} \int_Q |f(x) - \mathrm{av}_Q f| \, dx \right) < \infty,$$

where the sup ranges over all cubes $Q \subseteq R^n$, and $\mathrm{av}_Q f$ is the mean of f over Q. See [5]. For convenience, we identify f and f' in BMO if $f - f'$ is constant.

THEOREM 1. *BMO is the dual of the Hardy space* $H^1(R^n)$. *The inner product is given by* $\langle f, g \rangle = \int_{R^n} f(x) g(x) \, dx$ *for* $f \in BMO$ *and* g *belonging to the dense subspace of* C^∞ *rapidly decreasing functions in* H^1.

Here, we regard H^1 as the space of $f \in L^1(R^n)$ whose Riesz transforms $R_j(f)$ are all in L^1. See [7].

THEOREM 2. *A function belongs to BMO if and only if it can be written in the form* $g_0 + \sum_{j=1}^n R_j(g_j)$ *with* $g_0, g_1, \cdots, g_n \in L^\infty(R^n)$.

Note that the usual definition

$$R_j(g)(x) = \lim_{\epsilon \to 0; M \to \infty} \int_{\epsilon < |x-y| < M} K_j(x-y) f(y) \, dy$$

with $K_j(y) = c y_j / |y|^{n+1}$ need not make sense for all $g \in L^\infty$. (Consider $g(x) = \mathrm{sgn}(x)$ on the line.) Therefore, we define

$$R_j(g)(x) = \lim_{\epsilon \to 0} \int_{\epsilon < |x-y|} [K_j(x-y) - K_j^0(-y)] g(y) \, dy,$$

where $K_j^0(y) = K_j(y)$ for $|y| > 1$ and $K_j^0(y) = 0$ for $|y| \leqq 1$. This makes sense for all $g \in L^\infty$, and agrees with the usual definition up to an additive constant if g has compact support. See [3, p. 105].

The main idea in proving Theorems 1 and 2 is to study the Poisson integral of a function in BMO. Recall that any function f satisfying

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$$(*) \qquad \int_{R^n} \frac{|f(x)|}{(|x|+1)^{n+1}}\,dx < \infty$$

has a Poisson integral $u(x, t) = \text{P.I.}(f)$ defined on $R_+^{n+1} = R^n \times (0, \infty)$.

THEOREM 3. *A function f belongs to BMO if and only if* (*) *holds and* $\iint_{|x-x_0|<\delta;\ 0<t<\delta} t|\nabla u(x, t)|^2\,dx\,dt \leqq C\delta^n$ *for all* $x_0 \in R^n$ *and* $\delta > 0$.

Theorems 1–3 and their proofs can be used to study H^1. For example,

THEOREM 4. *Let* $F = (u_0(x, t); u_1(x, t), \cdots, u_n(x, t))$ *be an* $(n+1)$-*tuple of harmonic functions on* R_+^{n+1}, *satisfying the Cauchy-Riemann equations of* [7]. *If the nontangential maximal function* $u_0^*(x) \equiv \sup_{|x'|<t;\ t>0} |u_0(x-x', t)|$ *belongs to* L^1, *then F is in* H^1.

Different techniques enable us to replace L^1 and H^1 by L^p and H^p, $0 < p < \infty$. This generalizes a one-dimensional result of D. Burkholder, R. Gundy, and M. Silverstein (see [1] and [2]).

Further applications of Theorems 1–3 appear in [4] and [6]. [4] contains detailed proofs of the results stated here.

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