B(Top)_n~ AND THE SURGERY OBSTRUCTION

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This note announces “calculations” of the homotopy type of B(Top)_n~ and the nonsimply-connected surgery obstruction. Proofs, more precise statements, and consequences will appear in [6].

Remove the extraneous 2-torsion from KO by forming the pullback

\[
\begin{array}{c}
B_0^* \to \coprod_i (K(Z[1/\text{odd}], 4i)) \\
\downarrow \quad \downarrow \\
B_0 \otimes Z[\frac{1}{2}] \xrightarrow{ph} \coprod_i K(Q, 4i),
\end{array}
\]

and define

\[
L = B_0^* \times \prod_i K(Z/2, 4i + 2).
\]

L is a periodic multiplicative spectrum with product \(\otimes\) in \(B_0^*\), and cohomology multiplication in the \(Z/2\) part. \(B_0^*\) acts on the \(Z/2\) part by reduction mod 2, which gives \(\coprod_i K(Z/2, 4i)\), and inclusion in \(\prod_i K(Z/2, 2i)\).

Students of surgery will recognize Sullivan’s calculation in [7] as \(G/\text{TOP} \times Z \simeq L\). The Whitney sum in \(G/\text{TOP}\), however, is given by \(a \oplus b = a + b + 8a \otimes b\) in \(L\).

**Theorem 1.** Topological block bundles are naturally oriented in \(L\). If \(B_{L^n}\) is the classifying space for \(L\)-oriented \(G_n\) bundles, this induces a diagram of fibrations, for \(n \geq 3\).

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\(^2\) (Added in proof.) This cohomology structure was deduced using product formulas inferred from [7], [8]. This formula is now known to be wrong, and modified versions have been obtained by several groups. A slightly more complicated structure is thus required on \(L\), and will be corrected in [6].

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\[ \begin{array}{c}
G/\text{TOP} \\
\| \\
SG_n \to SG_n/\text{S(TOP}_n) \to B_{S(TOP}_n) \to B_{SG_n}
\end{array} \]

where \( Q = \prod_i [K(Z/8, 4i) \times K(Z/2, 4i+2) \times K(Z/2, 4i+3)] \), and \( L^* \) classifies the units of \( H^0(X; L) \).

\( L^* \cong G/\text{TOP} \). Thus a \( S(\text{TOP}_n) \sim \) bundle is an \( L \)-oriented \( G_n \) bundle, with a cohomology of the resulting cocycles \( q^{\epsilon} \in C^*(X; Z/8 \text{ and } Z/2) \) to zero. The Thom isomorphism comes from a "cobordism" interpretation of \( L \). The natural product in this interpretation is essentially \( \otimes \) in \( G/\text{TOP} \), hence \( 8\otimes \). Naturality shows the Thom isomorphism is multiplicative when taken \( \otimes Z[\frac{1}{2}] \). For \( Z[1/\text{odd}] \), the fact that \( \text{MSTOP} \) is a product of Eilenberg-Mac Lane spectra allows construction of the \( L \) Thom isomorphism from the one in topological cobordism. It is therefore multiplicative with respect to \( \otimes \), and is a product with a Thom class. \( Q \) is evaluated by showing \( G/\text{TOP} \cong L^* \to L^* \) is \( \alpha \mapsto 1 + 8(\alpha - 1) \).

This theorem, when taken \( \otimes Z[\frac{1}{2}] \), is

\[ B_{\text{TOP}_n} \sim B_{\text{KOG}_n}, \]

which has been announced by Sullivan [8]. The form of our result has been greatly influenced by Sullivan.

**Corollary.** If \( X \) is a simply-connected Poincaré space of dimension \( \geq 5 \) (\( \geq 6 \) if \( \partial X \neq \emptyset \), and then \( \pi_5\partial X = 0 \) also), then \( X \) has the homotopy type of a topological manifold if and only if it satisfies Poincaré duality in \( L \), and certain \( Z/8 \) and \( Z/2 \) characteristic homology classes of \( [X]_L \) vanish.

**Proof.** The SW dual of a fundamental class is a Thom class for the normal bundle \( \nu_X \). The homology characteristic classes are the ones which dualize to \( q^* \) of Theorem 1, so their vanishing implies \( \nu_X \) has a reduction to \( B_{\text{TOP}} \). Standard surgery now implies that \( X \) is homotopy equivalent to a manifold.

The different manifold structures on \( X \) correspond to liftings of \( \nu_X \) to \( B_{\text{TOP}} \) with zero surgery obstruction. The liftings may be specified as different \( L \) fundamental classes, together with homologies of the
q-cycles to zero. The vanishing of the surgery obstruction can be expressed as follows:

**Theorem 2.** Suppose $X$ is a Poincaré space of dimension $n \geq 5$ ($\geq 6$ if $\partial X \neq \emptyset$), with a reduction of $\nu X$ to $B_{\text{TOP}}$ which has surgery obstruction $\theta \in L_n(\pi_1 \partial X \to \pi_1 X)$, then the diagram commutes. Here the

$$
\begin{array}{ccc}
[X, G/\text{TOP}] & \xrightarrow{\sigma - \theta} & L_n(\pi_1 X \to \pi_1 X) \\
\cap & & \\
H^0(X; L) & \overset{A}{\longrightarrow} & L_n(\pi_1)
\end{array}
$$

inclusion is via $G/\text{TOP} \times Z \simeq L$, and $A$ is a universal homomorphism.

There is a similar diagram for boundary fixed $([X, \partial X; G/\text{TOP}, *] \to L_n(\pi_1 X))$ and for simple homotopy equivalences (just add superscript $s$ to $L_n$ and $A$). Note that if $\eta \in [X, G/\text{TOP}]$, $\eta \cap [X]_L$ is not the corresponding $L$ fundamental class for $X$, but "$\beta_1$" of it.

Julius Shaneson has pointed out that since $A$ is a homomorphism, and, for $\pi$ finite of odd order, $H_{\text{odd}}(K(\pi, 1); L)$ has odd order, and $L_{\text{odd}}(\pi)$ has exponent 4 $[3]$, $\sigma - \theta$ must be zero.

**Corollary.** The surgery obstruction of a normal map over a closed manifold of odd dimension and with $\pi$ finite of odd order is zero.

A much deeper proof for $\pi$ cyclic has been given by Browder [1].

The universal homomorphism $A$ may be used to obtain information on $L_n(\pi)$ in special cases. We define a class of groups we can treat.

If $G_1, G_2$ are groupoids, $f, g : G_1 \to G_2$ are homomorphisms, then the *generalized free product* of $G_2$ amalgamated over $f, g$ is given by: for a component $G_{1, a}$ of $G_1$, if $f, g$ map it into different components of $G_2$ take their free product and amalgamate over $f, g \vert G_{1, a}$. If $f, g$ map it into the same component of $G_2$, say $G_{2, a}$, form $G_{2, a} * J/N$, where $J$, infinite cyclic, is generated by $t$, and $N$ is generated by $f(x) = tg(x)t^{-1}$ for $x \in G_{1, a}$. Take a direct limit to get a groupoid.

$\pi$ is accessible of order 0 if each component is trivial, and accessible of order $n$ if it is a gfp with amalgamation, where the groupoids are accessible of order $n - 1$, and the amalgamating homomorphisms are all injective. This definition is due to Waldhausen [10], who shows that if $\pi$ is accessible of order 3 then $\text{Wh}(\pi) = 0$, and conjectures this
result for all accessible \( \pi \). An accessible group has a \( K(\pi, 1) \) of finite dimension.

Further, call \( \pi \) 2-sidedly accessible if each of the amalgamations is over 2-sided subgroups: \( H \subset G \) is 2-sided iff \( HxH = Hx^{-1}H \Rightarrow x \in H \) (e.g. all 2-torsion is in \( H \), see [2]). This condition arises in the codimension 1 splitting theorem of Cappell [2]. An early version of this theorem was applied in [4] (see also [5]) to obtain

**Theorem 3.** If \( \pi \) is 2-sidedly accessible, the universal homomorphism \( A : H_n(K(\pi, 1); L) \to L_n(\pi) \) has kernel and cokernel finite 2-groups.

The discrepancy comes from Wh(\( \pi' \)), \( \pi' \) in the construction of \( \pi \), so if Waldhausen’s conjecture is true, \( A \) is an isomorphism. In particular if \( \pi \) has order \( \leq 3 \), then \( A \) is an isomorphism.

**Corollary.** If \( \pi \) is free, free abelian, or a \( 3 \)-dimensional knot group, \( A : H_n(K(\pi, 1); L) \simeq L_n(\pi) \) is an isomorphism. In the last case abelianization \( L_n(\pi) \to L_n(Z) \) is an isomorphism.

**Proof.** These groups are accessible. That \( \pi_1(S^3 - C) \), \( C \) a curve, is accessible is due to Waldhausen [9], that it is of order 2 is in [10]. Abelianization \( \pi \to Z \) is a homology isomorphism, so since both \( L \) groups are homology groups, they are isomorphic.

Finally from the groups \( L_n(G) \) we can construct a spectrum \( L(G) \) with \( \pi_*L(G) = L_*((G) \). (\( L \) is \( L(0) \).) The same analysis as Theorem 3 gives

**Theorem 4.** There is a natural homomorphism \( A : H_n(K(\pi, 1); L(G)) \to L_n(\pi \times G) \), which has kernel and cokernel finite 2-groups if \( \pi \) is 2-sidedly accessible.

This calculation generalizes (up to 2-groups) that of Shaneson for \( G \times Z \). For modest \( \pi \) (e.g. \( Z \)) the 2-groups can be kept track of. Finally extensions \( 1 \to G_1 \to G_2 \to \pi \to 1 \) can be described as homology of \( K(\pi, 1) \) with twisted coefficients in \( L(G_1) \).

**References**


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