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$B_{(\mathrm{TOP}_n)^\sim}$ AND THE SURGERY OBSTRUCTION[1]

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This note announces "calculations" of the homotopy type of $B_{(\mathrm{TOP}_n)^\sim}$ and the nonsimply-connected surgery obstruction. Proofs, more precise statements, and consequences will appear in [6].

Remove the extraneous 2-torsion from KO by forming the pullback

$$\begin{array}{ccc} B_0^* & \longrightarrow & \prod_i (K(Z[1/\mathrm{odd}], 4i)) \\ \downarrow & & \downarrow \\ B_0 \otimes Z[\frac{1}{2}] & \xrightarrow{ph} & \prod_i K(Q, 4i), \end{array}$$

and define

$$L = B_0^* \times \prod_i K(Z/2, 4i+2).$$

L is a periodic multiplicative spectrum with product $\otimes$ in B_0^*, and cohomology multiplication in the $Z/2$ part. B_0^* acts on the $Z/2$ part by reduction mod 2, which gives $\prod_i K(Z/2, 4i)$, and inclusion in $\prod_i K(Z/2, 2i)$.[2]

Students of surgery will recognize Sullivan's calculation in [7] as $G/\mathrm{TOP} \times Z \simeq L$. The Whitney sum in G/TOP, however, is given by $a \oplus b = a + b + 8a \otimes b$ in L.

THEOREM 1. *Topological block bundles are naturally oriented in* L. *If* B_{LG_n} *is the classifying space for* L*-oriented* G_n *bundles, this induces a diagram of fibrations, for* $u \geqq 3$,

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[2] (ADDED IN PROOF.) This cohomology structure was deduced using product formulas inferred from [7], [8]. This formula is now known to be wrong, and modified versions have been obtained by several groups. A slightly more complicated structure is thus required on L, and will be corrected in [6].

$$
\begin{array}{ccccccc}
 & & G/\mathrm{TOP} & & & & \\
 & & \| & & & & \\
SG_n & \longrightarrow & SG_n/S(\mathrm{TOP}_n)^\sim & \longrightarrow & B_{S(\mathrm{TOP}_n)^\sim} & \longrightarrow & B_{SG_n} \\
\| & & \downarrow & & \downarrow & & \downarrow \| \\
SG_n & \longrightarrow & \boldsymbol{L}^* & \longrightarrow & B_{LG_n} & \longrightarrow & B_{SG_n} \\
 & & \downarrow & & \downarrow & & \\
 & & Q & = & Q & &
\end{array}
$$

where $Q=\prod_i [K(Z/8, 4i)\times K(Z/2, 4i+2)\times K(Z/2, 4i+3)]$, *and* $\boldsymbol{L}^*$ *classifies the units of* $H^0(X; \boldsymbol{L})$.

$\boldsymbol{L}^*\simeq G/\mathrm{TOP}$. Thus a $S(\mathrm{TOP}_n)^\sim$ bundle is an $\boldsymbol{L}$-oriented G_n bundle, with a cohomology of the resulting cocycles $q^\# \in C^*(X; Z/8$ and $Z/2)$ to zero. The Thom isomorphism comes from a "cobordism" interpretation of $\boldsymbol{L}$. The natural product in this interpretation is essentially $\oplus$ in G/TOP, hence $8\otimes$. Naturality shows the Thom isomorphism is multiplicative when taken $\otimes Z[\frac{1}{2}]$. For $Z[1/\mathrm{odd}]$, the fact that MSTOP is a product of Eilenberg-Mac Lane spectra allows construction of the $\boldsymbol{L}$ Thom isomorphism from the one in topological cobordism. It is therefore multiplicative with respect to $\otimes$, and is a product with a Thom class. Q is evaluated by showing $G/\mathrm{TOP}\simeq \boldsymbol{L}^*\rightarrow \boldsymbol{L}^*$ is $a\mapsto 1+8(a-1)$.

This theorem, when taken $\otimes Z[\frac{1}{2}]$, is

$$B_{(\mathrm{TOP}_n)^\sim} \simeq B_{KOG_n},$$

which has been announced by Sullivan [8]. The form of our result has been greatly influenced by Sullivan.

COROLLARY. *If X is a simply-connected Poincaré space of dimension* $\geqq 5$ ($\geqq 6$ *if* $\partial X\neq\emptyset$, *and then* $\pi_1\partial X=0$ *also*), *then X has the homotopy type of a topological manifold iff it satisfies Poincaré duality in* $\boldsymbol{L}$, *and certain $Z/8$ and $Z/2$ characteristic homology classes of* $[X]_L$ *vanish.*

PROOF. The SW dual of a fundamental class is a Thom class for the normal bundle ν_X. The homology characteristic classes are the ones which dualize to q^* of Theorem 1, so their vanishing implies ν_X has a reduction to B_{TOP}. Standard surgery now implies that X is homotopy equivalent to a manifold.

The different manifold structures on X correspond to liftings of ν_X to B_{TOP} with zero surgery obstruction. The liftings may be specified as different $\boldsymbol{L}$ fundamental classes, together with homologies of the

q-cycles to zero. The vanishing of the surgery obstruction can be expressed as follows:

THEOREM 2. *Suppose X is a Poincaré space of dimension $n \geqq 5$ ($\geqq 6$ if $\partial X \neq \varnothing$), with a reduction of ν_X to B_{TOP} which has surgery obstruction $\theta \in L_n(\pi_1 \partial X \to \pi_1 X)$, then the diagram commutes. Here the*

$$\begin{array}{ccc}
[X, G/\mathrm{TOP}] & \xrightarrow{\sigma-\theta} & L_n(\pi_1 X \to \pi_1 X) \\
\cap\!\mid & & \uparrow \\
H^0(X; \boldsymbol{L}) & & \Big| A \\
\wr\!\mid \cap [X]_{\boldsymbol{L}} & & \Big| \\
H_n(X, \partial X; \boldsymbol{L}) & \xrightarrow[H_n(\pi_1)]{} & H_n(K(\pi_1 X, 1), K(\pi_1 Y, 1); \boldsymbol{L})
\end{array}$$

inclusion is via $G/\mathrm{TOP} \times Z \simeq \boldsymbol{L}$, and A is a universal homomorphism.

There is a similar diagram for boundary fixed ($[X, \partial X; G/\mathrm{TOP}, *] \to L_n(\pi_1 X)$) and for simple homotopy equivalences (just add superscript s to L_n and A). Note that if $\eta \in [X, G/\mathrm{TOP}]$, $\eta \cap [X]_L$ is not the corresponding $\boldsymbol{L}$ fundamental class for X, but "$\frac{1}{8}$" of it.

Julius Shaneson has pointed out that since A is a homomorphism, and, for π finite of odd order, $H_{\mathrm{odd}}(K(\pi, 1); \boldsymbol{L})$ has odd order, and $L_{\mathrm{odd}}(\pi)$ has exponent 4 [**3**], $\sigma - \theta$ must be zero.

COROLLARY. *The surgery obstruction of a normal map over a closed manifold of odd dimension and with π finite of odd order is zero.*

A much deeper proof for π cyclic has been given by Browder [**1**].

The universal homomorphism A may be used to obtain information on $L_n(\pi)$ in special cases. We define a class of groups we can treat.

If G_1, G_2 are groupoids, f, $g: G_1 \to G_2$ are homomorphisms, then the *generalized free product* of G_2 amalgamated over f, g is given by: for a component $G_{1,\alpha}$ of G_1, if f, g map it into different components of G_2 take their free product and amalgamate over f, $g | G_{1,\alpha}$. If f, g map it into the same component of G_2, say $G_{2,\alpha}$, form $G_{2,\alpha} * J/N$, where J, infinite cyclic, is generated by t, and N is generated by $f(x) = tg(x)t^{-1}$ for $x \in G_{1,\alpha}$. Take a direct limit to get a groupoid.

π is *accessible of order* 0 if each component is trivial, and *accessible of order n* if it is a gfp with amalgamation, where the groupoids are accessible of order $n-1$, and the amalgamating homomorphisms are all injective. This definition is due to Waldhausen [**10**], who shows that if π is accessible of order 3 then $\mathrm{Wh}(\pi) = 0$, and conjectures this result for all accessible π. An accessible group has a $K(\pi, 1)$ of finite dimension.

Further, call π *2-sidedly accessible* if each of the amalgamations is over 2-sided subgroups: $H \subset G$ is *2-sided* iff $HxH = Hx^{-1}H \Rightarrow x \in H$ (e.g. all 2-torsion is in H), see [**2**]. This condition arises in the codimension 1 splitting theorem of Cappell [**2**]. An early version of this theorem was applied in [**4**] (see also [**5**]) to obtain

THEOREM 3. *If π is 2-sidedly accessible, the universal homomorphism $A: H_n(K(\pi, 1); \boldsymbol{L}) \rightarrow L_n(\pi)$ has kernel and cokernel finite 2-groups.*

The discrepancy comes from $\mathrm{Wh}(\pi')$, π' in the construction of π, so if Waldhausen's conjecture is true, A is an isomorphism. In particular if π has order $\leqq 3$, then A is an isomorphism.

COROLLARY. *If π is free, free abelian, or a 3-dimensional knot group, $A: H_n(K(\pi, 1); \boldsymbol{L}) \simeq L_n(\pi)$ is an isomorphism. In the last case abelianization $L_n(\pi) \rightarrow L_n(Z)$ is an isomorphism.*

PROOF. These groups are accessible. That $\pi_1(S^3 - C)$, C a curve, is accessible is due to Waldhausen [**9**], that it is of order 2 is in [**10**]. Abelianization $\pi \rightarrow Z$ is a homology isomorphism, so since both L groups are homology groups, they are isomorphic.

Finally from the groups $L_n(G)$ we can construct a spectrum $\boldsymbol{L}(G)$ with $\pi_* \boldsymbol{L}(G) = \boldsymbol{L}_*(G)$. ($\boldsymbol{L}$ is $\boldsymbol{L}(0)$.) The same analysis as Theorem 3 gives

THEOREM 4. *There is a natural homomorphism $A: H_n(K(\pi, 1); \boldsymbol{L}(G)) \rightarrow L_n(\pi \times G)$, which has kernel and cokernel finite 2-groups if π is 2-sidedly accessible.*

This calculation generalizes (up to 2-groups) that of Shaneson for $G \times Z$. For modest π (e.g. Z) the 2-groups can be kept track of. Finally extensions $1 \rightarrow G_1 \rightarrow G_2 \rightarrow \pi \rightarrow 1$ can be described as homology of $K(\pi, 1)$ with twisted coefficients in $\boldsymbol{L}(G_1)$.

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