SUBSTITUTION MINIMAL FLOWS

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We investigate the structure of a certain class of minimal symbolic flows (substitution minimal flows) which are natural generalizations of the widely studied Morse minimal set (see, for example, [3], [5]). We present here a brief description of the major results; detailed proofs will appear elsewhere. The author wishes to thank William Veech for his help in the preparation of this paper.

Let $S = \{0, 1, \cdots, b - 1\}$, and for $n \geq 1$, let $S^n = \{f: \{0, 1, \cdots, n-1\} \rightarrow S\}$. If $A \in S^n$, we represent $A$ as $a_0 \cdots a_{n-1}$, where $a_i = A(i)$; we refer to $A$ as an $n$-block (over $S$). For $A \in S^n$, $B \in S^m$, we let $AB = a_0a_1 \cdots a_{n-1}b_1b_2 \cdots b_{m-1}$, so that $AB \in S^{n+m}$. A substitution $\theta (= \theta^1)$ of length $r$ over $S$ is a map $\theta: S \rightarrow S^r$ with $\theta(0)(0) = 0$. For $k \geq 2$, if $\theta(j) = a_0a_1 \cdots a_{r-1}$, we define $\theta^k(j) = \theta^{k-1}(a_0) \cdots \theta^{k-1}(a_{r-1})$. We define a sequence $x_\theta^r$ over $S$ by letting the $r^r$-block $x_\theta^r(0)x_\theta^r(1) \cdots x_\theta^r(r-1)$ be $\theta^k(0)$, for each $k \geq 1$. $\theta$ is an admissible substitution if $\theta$ is one-to-one, range $x_\theta^r = S$, and $x_\theta^r$ is a recurrent, nonperiodic sequence. (It is not difficult to prescribe simple conditions which ensure that $\theta$ is admissible.) $\theta$ is simple, if for $i, j \in S$ ($i \neq j$), $\theta(i)(n) \neq \theta(j)(n)$ ($0 \leq n \leq r - 1$). If $\theta$ is an admissible substitution, we choose any recurrent extension $x_\theta$ of $x_\theta^r$ to the integers, and we define $X_\theta = (X_\theta, T)$ to be the flow whose phase space $X_\theta$ is the orbit-closure of $x_\theta$ under the left shift $T$, in the space of all doubly infinite sequences over $S$ (with the product topology). $X_\theta$ is an infinite, compact metric space, and $X_\theta$ is a minimal flow. Finally, we obtain a positive integer $m(\theta)$ with $\gcd(m(\theta), r) = 1$ so that $S$ is partitioned into nonempty sets $S_0, S_1, \cdots, S_{m(\theta)-1}$, and if $i \in S_{\theta(i)}$ ($i \in S$), the sequence of integers $n(x_\theta(j))$ ($j = 0, 1, \cdots$) is periodic of period $m(\theta)$.

If $\theta$ is a fixed admissible substitution of length $r$ over $S$, our principal results may be stated as follows. Some of our results generalize certain results in [1] and [4]. (All definitions are as in [10].)
THEOREM 1. $X_\theta$ is a point-distal flow with a residual set of distal points.

THEOREM 2. Let $\Sigma$ be the equicontinuous structure relation on $X_\theta$. Then $X_\theta/\Sigma$ is isomorphic to the equicontinuous flow $(\mathbb{Z}_{m(\theta)} \times \mathbb{Z}_r, T)$, where $\mathbb{Z}_{m(\theta)}$ is the cyclic group of order $m(\theta)$, $\mathbb{Z}_r$ is the $r$-adic completion of the integers, and $T$ is the homeomorphism determined by addition of the group element $(1, 1)$.

COROLLARY. If $\theta$ is a binary substitution, $X_\theta/\Sigma = (\mathbb{Z}_r, T)$.

THEOREM 3. $X_\theta$ is an almost automorphic flow if and only if there exist integers $i, j, k (0 \leq i \leq m(\theta) - 1, j \geq 1, 0 \leq k \leq r^j - 1)$ with $\theta_i(p)(k) = \theta^j(q)(k)$ for $p, q \in S_i$.

In [8], Veech represents the Morse flow (the substitution flow generated by the binary substitution $\theta(0) = 01, \theta(1) = 10$) as an isometric extension of an almost automorphic extension of $(\mathbb{Z}_2, T)$. This may be generalized in the following manner. We define $P_\theta = \{x_\theta(j)x_\theta(j+1): j = 0, 1, \cdots \} \subset S^2; A_{ijk} = \{\theta(p)(k)\theta^j(q)(k+1): p \in S_i \}

THEOREM 4. If $\theta$ is simple, $X_\theta$ is an AI extension (i.e., an isometric extension of an almost automorphic extension) of an equicontinuous flow if and only if the collection $\{A_{ijk}\}$ is a partition of $P_\theta$.

It can easily be seen that this condition holds automatically for every simple binary substitution. We obtain

THEOREM 5. If $\theta$ is a binary substitution of length $r$, $X_\theta$ is either an almost automorphic flow or an AI extension of the equicontinuous flow $(\mathbb{Z}_r, T)$.

THEOREM 6. If $\theta$ is simple, and $r$ and $b$ are both prime, $X_\theta$ is an AI flow if and only if the collection $\{A_{ijk}\}$ is a partition of $P_\theta$.

By Theorem 6, we obtain a class of point-distal flows with a residual set of distal points which are not AI flows. This is significant in the light of Veech's structure theorem for point-distal flows [10], according to which every point-distal flow with a residual set of distal points has an almost automorphic extension which is an AI flow. (Leonard Shapiro, in [6], has constructed examples, of a different sort, of point-distal, non-AI flows.)

EXAMPLE. Let $b = r = 3; \theta(0) = 011, \theta(1) = 202, \theta(2) = 120$. It can be easily verified that $\theta$ is admissible and simple and that $m(\theta) = 1$. We have $A_{100} \cap A_{011} = \{20\}$, and thus, by Theorem 6, $X_\theta$ is not an AI flow.
We remark that for substitutions of nonconstant length (i.e., if the blocks \( \theta(0), \theta(1), \ldots, \theta(b-1) \) are not of the same length), the situation is substantially different. \( \mathcal{X}_\theta \) is no longer point-distal in general, and for certain \( \theta \), \( \mathcal{X}_\theta \) can be shown to be weakly mixing. We hope to discuss this at greater length in a later paper.

References


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