BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 4, July 1971

COMPUTATION OF SYMBOLS ON C^*-ALGEBRAS OF SINGULAR INTEGRAL OPERATORS[1]

BY R. D. MOYER

Communicated by I. M. Singer, January 7, 1971

In recent years various authors have studied C^*-algebras generated by singular integral operators, Wiener-Hopf operators, Toeplitz operators, etc., and representations of these algebras. Analysis in these algebras is accomplished modulo a suitable ideal (usually the commutator and/or compact ideal). A similar procedure is used in the theory of pseudo-differential operators except that the emphasis is more on the asymptotic behavior of the operators (see [8]). We would like to discuss an adaptation of this asymptotic analysis to some C^*-algebras.

In what follows, H will be a Hilbert space and $\mathcal{L}(H)$ its endomorphism algebra. Let X be a set of nets of automorphisms[2] of $\mathcal{L}(H)$. If $x \in X$, then

$$\mathfrak{A}_x = \{A \in \mathcal{L}(H): x(A) \text{ has a strong-* limit}\}$$

is a C^*-algebra, and hence so is $\mathfrak{A}_X = \bigcap_x \mathfrak{A}_x$. Define the map $\sigma: \mathfrak{A}_X \to \mathcal{L}(H)^X$ (the functions from X to $\mathcal{L}(H)$) by

$$\sigma_x(A) = \lim x(A), \qquad A \in \mathfrak{A}_X,\ x \in X.$$

σ is a *-homomorphism and will be called the *symbol* of the algebra $\mathfrak{A}_X$.

Consider the special case of a set X such as above and an C^*-algebra $\mathfrak{B}$ generated by two C^*-algebras $\mathfrak{M}$ and $\mathfrak{S}$ in $\mathfrak{A}_X$. Assume that for each $x \in X$,

(i) $\sigma_x = \text{id}$ on $\mathfrak{M}$,

(ii) $\sigma_x(\mathfrak{S}) \subset \mathbf{C}I$.

Let $\mathcal{B}(X)$ denote the algebra of bounded functions on X equipped with sup norm. We shall canonically identify $\mathfrak{M} \otimes \mathcal{B}(X)$ with a C^*-subalgebra of $\mathcal{L}(H)^X$. Under assumptions (i) and (ii), σ maps $\mathfrak{M}$ isomorphically onto $\mathfrak{M} \otimes \mathbf{C}$ and $\mathfrak{S}$ into $\mathbf{C} \otimes \mathcal{B}(X)$.

AMS 1970 *subject classifications*. Primary 45E10, 47G05; Secondary 35S05, 58G15.
Key words and phrases. Symbols, singular integral operators, Wiener-Hopf operators, Toeplitz operators, index theorems.

[1] This work was supported in part by the National Science Foundation.

[2] More generally, take representations of a fixed C^*-algebra.

PROPOSITION. *If* $\mathfrak{M}$ *is commutative and* $\ker\sigma\cap\mathfrak{S}=\{0\}$,[3] *then* $\ker\sigma$ *is the commutator ideal in* $\mathfrak{B}$.

PROOF. $\sigma(\mathfrak{B})=\mathfrak{M}\otimes\sigma(\mathfrak{S})$, a commutative C^*-algebra. Since σ is an isomorphism on $\mathfrak{S}$, $\mathfrak{S}$ is commutative and σ induces a homeomorphism of the maximal ideal space of $\sigma(\mathfrak{B})$ with the product of those of $\mathfrak{M}$ and $\mathfrak{S}$. But then every character on $\mathfrak{B}$ is determined on $\mathfrak{M}$ and $\mathfrak{S}$ by evaluating σ at a maximal ideal of $\sigma(\mathfrak{B})$, and the result follows.

EXAMPLES. (1) Embed $C(S^{n-1})$ into $\mathfrak{L}^\infty(\boldsymbol{R}^n)$ by extending each φ in $C(S^{n-1})$ to a function on $\boldsymbol{R}^n$ homogeneous of degree 0. We shall consider $\mathfrak{L}^\infty(\boldsymbol{R}^n)$ to be represented as multiplication operators on $H=\mathfrak{L}^2(\boldsymbol{R}^n)$. Let $\mathfrak{F}$ denote Fourier transform, and for $\lambda\in\boldsymbol{R}^n$ let e_λ be the function $x\mapsto e^{i\lambda\cdot x}$. If $\varphi\in C(S^{n-1})$ and $\lambda\in S^{n-1}$,

$$e_{-t\lambda}\mathfrak{F}^{-1}\varphi\mathfrak{F}e_{t\lambda}\to\varphi(\lambda)I$$

strongly as $t\to\infty$. Hence, if we take X to be the set of nets of inner automorphisms induced by the unitary semigroups $\{e_{t\lambda}\}_{t>0}$, $\lambda\in S^{n-1}$, and set $\mathfrak{M}=\mathfrak{L}^\infty(\boldsymbol{R}^n)$ and $\mathfrak{S}=\mathfrak{F}^{-1}C(S^{n-1})\mathfrak{F}$, the algebra $\mathfrak{B}$ has the symbol

$$\sigma_\lambda(B)=\lim_{t\to\infty}e_{-t\lambda}Be_{t\lambda}\in\mathfrak{L}^\infty(\boldsymbol{R}^n),\qquad\lambda\in S^{n-1}. \tag{1}$$

Clearly, σ induces an isomorphism of $\mathfrak{S}$ on $\boldsymbol{C}\otimes C(S^{n-1})$ and $\sigma=\mathrm{id}$ on $\mathfrak{L}^\infty(\boldsymbol{R}^n)$. Therefore, σ is a *-homomorphism of $\mathfrak{B}$ onto $\mathfrak{L}^\infty(\boldsymbol{R}^n)\otimes C(S^{n-1})$ whose kernel is the commutator ideal in $\mathfrak{B}$. Restricting $\mathfrak{M}$ to various subalgebras of $\mathfrak{L}^\infty(\boldsymbol{R}^n)$, one obtains an explicit method of computing the symbols defined by Gohberg [6], Seeley [10], Cordes [4], Newberger [9], Herman [7], Cordes and Herman [5]. See also [3]. Unlike each of these cases where the compactly supported operators with symbol 0 are compact in $\mathfrak{L}(H)$, the algebra here does not have this property. Since local diffeomorphisms applied to operators in $\mathfrak{B}$ with suitable support give an operator in $\mathfrak{B}$ (this is true on $\mathfrak{L}^\infty(\boldsymbol{R}^n)$ and the usual singular integral operators), $\mathfrak{B}$ can be used as a modelling algebra for what should be called singular integral operators on compact manifolds with bounded measurable coefficients.[4] Analysis is no longer performed modulo the algebra of compact operators, but modulo a larger subalgebra of $\mathfrak{L}(H)$ whose structure is relatively unknown. For the moment let us simply note that it contains every compactly supported pseudo-differential operator of order less than 0 and every compact operator.

[3] It suffices to assume that this is the commutator ideal in $\mathfrak{S}$.

[4] The symbol of these operators can be computed directly on the manifold; see [8].

(2) Let G be a LCA group, $\hat{G}$ its character group, and Σ a subsemigroup of $\hat{G}$. Σ is directed by the natural ordering: $a \geqq b$ iff $a \in b\Sigma$. Choose a measurable set S in $\hat{G}$, with characteristic function χ_S, such that the nets $\{\chi_{\gamma S}\}_{\gamma\in\Sigma}$ and $\{\chi_{\gamma S}\}_{\gamma\in\Sigma^{-1}}$, considered as multiplication operators on $\mathcal{L}^2(\hat{G})$, converge strongly to 0 and 1, respectively. Let $\mathfrak{F}$ denote the Fourier-Plancherel transform on $\mathcal{L}^2(G)$, and let $P = \mathfrak{F}^{-1}\chi_S\mathfrak{F}$. Here we shall take X to be the pair

$$+ = \{A \mapsto \gamma^{-1}A\gamma\}_{\gamma\in\Sigma},$$

$$- = \{A \mapsto \gamma A\gamma^{-1}\}_{\gamma\in\Sigma}.$$

Also take $\mathfrak{M} = \mathcal{L}^\infty(G)$ and $\mathfrak{S} = \boldsymbol{C}I + \boldsymbol{C}P$. Then

$$\sigma_+(A) = \lim_\gamma \gamma^{-1}A\gamma,$$

$$\sigma_-(A) = \lim_\gamma \gamma A\gamma^{-1}$$

defines a *-homomorphism of $\mathfrak{S}$ onto $\mathcal{L}^\infty(G)^2$ whose kernel is the commutator ideal. In the spirit of Coburn and Douglas [1] we shall call $\mathfrak{B}$ the algebra of generalized singular integral operators on G relative to S. The generalized Wiener-Hopf or Toeplitz operators are the elements of the reduced algebra $P\mathfrak{B}P$. Thus, analysis of Wiener-Hopf operators can be accomplished from this point of view. In particular, σ_+ is the usual symbol of such operators and those with invertible symbols are invertible modulo ker σ_+. This does not completely answer the question posed in [1] since we do not know if ker σ_+ is the commutator ideal in the reduced algebra.

(3) As a combination of Examples 1 and 2, consider a proper open cone Γ in $\boldsymbol{R}^n$. Let $C_\Gamma(S^{n-1})$ be the set of functions in $\mathcal{L}^\infty(\boldsymbol{R}^n)$ which vanish on $\boldsymbol{R}^n\backslash\Gamma$, are uniformly continuous on Γ, and are homogeneous of degree 0. Here X will be the nets of inner automorphisms induced by $\{e_{t\lambda}\}_{t>0}$ for $\lambda\in\Gamma\cap S^{n-1}$. Then for $\mathfrak{M} = C_0(\boldsymbol{R}^n) + \boldsymbol{C}I$ and $\mathfrak{S} = \mathfrak{F}^{-1}C_\Gamma(S^{n-1})\mathfrak{F}$, $\mathfrak{B}$ has symbol σ given by (1) for $\lambda \in S^{n-1}\cap\Gamma$. As before, σ maps $\mathfrak{B}$ onto $[C_0(\boldsymbol{R}^n) + \boldsymbol{C}I] \otimes C_\Gamma(S^{n-1})$ with the commutator ideal as its kernel. It can be shown that the compactly supported operators with zero symbol are compact operators in this case.

(4) Let $\mathfrak{M}$ be the subalgebra of $\mathcal{L}^\infty(\boldsymbol{R}^n)$ of functions with a limit at ∞. Let $\mathfrak{D}$ be the algebra of all ψ in $C(\boldsymbol{R}^n)$ such that for some $\psi_\infty \in C(S^{n-1})$, $\psi - \psi_\infty$ vanishes at ∞, and set $\mathfrak{S} = \mathfrak{F}^{-1}\mathfrak{D}\mathfrak{F}$. Consider the homotheties

$$\eta_t u(x) = e^{nt/2}u(e^t x), \qquad x \in \boldsymbol{R}^n,\ t > 0.$$

Let X_1 be the set of automorphism semigroups

$$t \mapsto \eta_{-t} e_{-\lambda} A e_{\lambda} \eta_t, \qquad \lambda \in \boldsymbol{R}^n,$$

and let X_2 be the set of automorphism semigroups

$$t \mapsto e_{-t\lambda} A e_{t\lambda}, \qquad \lambda \in S^{n-1},$$

which were considered in Example 1. As before, these are nets with $\boldsymbol{R}_+$ directed toward ∞. We shall identify X_1 with $\boldsymbol{R}^n$ and X_2 with S^{n-1}. A straightforward computation gives for $\varphi \in \mathfrak{M}$ and $\psi \in \mathfrak{D}$,

$$\begin{aligned} \sigma_\lambda(\varphi) &= \varphi(\infty) I, & \lambda &\in X_1, \\ &= \varphi, & \lambda &\in X_2, \\ \sigma_\lambda(\mathfrak{F}^{-1}\psi\mathfrak{F}) &= \psi(\lambda) I, & \lambda &\in X_1, \\ &= \psi_\infty(\lambda) I, & \lambda &\in X_2. \end{aligned}$$

Thus σ is a *-homomorphism of $\mathfrak{B}$ into $[\mathfrak{M} \otimes \mathfrak{D}] \oplus [\mathfrak{M} \otimes C(S^{n-1})]$. The range of σ can be shown to be those pairs in $[\boldsymbol{C} \otimes \mathfrak{D}] \oplus [\mathfrak{M} \otimes C(S^{n-1})]$ which have compatible behavior at ∞. Also, ker σ is the commutator ideal, but this is obtained by modifying the idea in the proof of the Proposition. The essential point is that operators of the form $\varphi\mathfrak{F}^{-1}\psi\mathfrak{F}$ where *both* φ and ψ vanish at ∞ are compact. Since the algebra is irreducible, the commutator ideal contains the compact operators. Also, every character on $\mathfrak{B}$ must vanish either on the compactly supported elements of $\mathfrak{M}$ or on those in $\mathfrak{S}$. It is then easy to see that characters on $\mathfrak{B}$ are given by evaluation of σ. In the special case where $\mathfrak{M}$ is the continuous functions with limit at ∞, $\mathfrak{B}$ is one of the algebras considered by Cordes and Herman [**5**]. There ker σ is the compact ideal but that is not the case here.

(5) Let M_m denote the $m \times m$ complex matrices, and X some set of nets of automorphisms of $\mathfrak{L}(H)$. The symbol on $\mathfrak{A}_X$ has a natural extension to $\mathfrak{A}_X \otimes M_m$ by replacing X by $X \otimes I$. In fact $\mathfrak{A}_{X\otimes I} = \mathfrak{A}_X \otimes M_m$. Thus, systems need not be introduced in an ad hoc fashion.

(6) This last example differs from the main theme in that the symbol is defined by considering a finite succession of limits of one parameter semigroups of automorphisms. More precisely, for $n \geqq 2$, we shall inductively define the algebra $R(S^{n-1})$ to be the algebra of all bounded measurable functions ψ on $\boldsymbol{R}^n$ which are homogeneous of degree 0 and have the property that for each $\lambda \in \boldsymbol{R}^n \setminus \{0\}$, there exists a $\psi_\lambda \in R(S^{n-2})$ such that if P_λ is the orthogonal projection of $\boldsymbol{R}^n$ along λ onto $\boldsymbol{R}^{n-1}$, $\psi(\epsilon x + \lambda) - \psi_\lambda(P_\lambda x)$ tends to 0 in $\mathfrak{L}^1$ on $\{x : |x| \leqq 1\}$ as $\epsilon \to 0$. Note that ψ_λ does not depend on $|\lambda|$. When

$n=1$, take $R(S^0)$ to be the functions constant on each half-line. When $n=2$, $R(S^1)$ contains the set of homogeneous functions with simple discontinuities on S^1. Take $\mathfrak{S}=\mathfrak{F}^{-1}R(S^{n-1})\mathfrak{F}$, and observe that, for any $\psi\in R(S^{n-1})$ and $\lambda\in \boldsymbol{R}^n\setminus\{0\}$, $e_{-t\lambda}\mathfrak{F}^{-1}\psi\mathfrak{F}e_{t\lambda}$ converges strongly in $\mathfrak{L}^2(\boldsymbol{R}^n)$ to $\mathfrak{F}^{-1}(\psi_\lambda \circ P_\lambda)\mathfrak{F}$. Also, it can be easily seen that $\psi_\lambda \circ P_\lambda \in R(S^{n-1})$. Therefore, if X is the set of nets of inner automorphisms induced by $\{e_{t\lambda}\}_{t>0}$, $\lambda\in\boldsymbol{R}^n$, and $\mathfrak{M}=\mathfrak{L}^\infty(\boldsymbol{R}^n)$, then $\mathfrak{B}$ has a symbol σ given by

$$\sigma_\lambda(B) = \lim_{t\to\infty} e_{-t\lambda}Be_{t\lambda}, \qquad \lambda \in \boldsymbol{R}^n.$$

The essential difference from the previous examples is that $\sigma_\lambda(B)$ in general is not in $\boldsymbol{C}I$, but in $\mathfrak{B}$ and the commutant of $\{e_{t\lambda}\}_{t\in\boldsymbol{R}}$. However, if $\lambda_1, \cdots, \lambda_n$ is any ordered orthonormal basis of $\boldsymbol{R}^n$, $\sigma_{\lambda_1}\circ \cdots \circ\sigma_{\lambda_n}$ maps $\mathfrak{S}$ into $\boldsymbol{C}I$ and hence $\sigma_{\lambda_1}\circ \cdots \circ\sigma_{\lambda_n}$ maps $\mathfrak{B}$ into $\mathfrak{M}$. Now, it can be shown that if $S\in\mathfrak{S}$ and $\sigma_\lambda(S)=0$ for every $\lambda\in S^{n-1}$, then $S=0$. Also, $\sigma_\lambda\circ\sigma_\mu=\sigma_{\lambda+\alpha\mu}\circ\sigma_\mu$ for every λ, $\mu\in\boldsymbol{R}^n$ and $\alpha\in\boldsymbol{R}$.

Therefore, if $S\in\mathfrak{S}$ and $\sigma_{\lambda_1}\circ \cdots \circ\sigma_{\lambda_n}(S)=0$ for every orthonormal basis $\lambda_1, \cdots, \lambda_n$, $S=0$. Applying the Proposition we see that viewed as a function on the set of ordered orthonormal bases of $\boldsymbol{R}^n$, the above defines a generalized symbol whose kernel is the commutator ideal.

REMARK. In general, $\sigma_{\lambda_1}\circ \cdots \circ\sigma_{\lambda_n}$ depends on the order of $\lambda_1, \cdots, \lambda_n$. For example, let $n=2$, let λ_1, λ_2 be the usual basis and let ψ be the characteristic function of the sector between λ_1 and $\lambda_1+\lambda_2$ (with angle $\pi/4$). Then $\sigma_{\lambda_1}(\psi)$ is the characteristic function of the upper half-plane and $\sigma_{\lambda_2}\circ\sigma_{\lambda_1}(\psi)=+1$. However, $\sigma_{\lambda_2}(\psi)=0$.

In closing we would like to comment on the recent index theorem of Coburn, Douglas, Schaeffer, and Singer [**2**]. In defining the analytical index for their operator algebra they embedded their algebra in a type II_∞ factor and employed the dimension function to obtain the index. It is not too difficult to see that the symbol on their algebra can be computed in a manner similar to Example 2. Moreover, the automorphism semigroups used to compute the symbol also define a symbol on the representation space and the correspondence between symbols is a *-isomorphism. This gives a C^*-algebra with symbol in a type II_∞ factor. Their index theorem states in part that the topological and analytical indices agree on this algebra. Since this is shown to be independent of the representation, one can interpret this to be an index theorem for the isomorphism class of the C^*-algebra together with its symbol.

References

1. L. A. Coburn and R. G. Douglas, *On C*-algebras of operators on a half-space.* I (to appear).

2. L. A. Coburn, R. G. Douglas, D. G. Schaeffer and I. M. Singer, *C*-algebras of operators on a half-space.* II: *Index theory* (to appear).

3. L. A. Coburn, R. D. Moyer and I. M. Singer, *Algebras of singular integral operators with almost periodic coefficients* (to appear).

4. H. O. Cordes, *The algebra of singular integral operators in* R^n, J. Math. Mech. **14** (1965), 1007–1032. MR **31** #6137.

5. H. O. Cordes and E. A. Herman, *Gel'fand theory of pseudo-differential operators*, Amer. J. Math. **90** (1968), 681–717.

6. I. C. Gohberg, *On the theory of multidimensional singular integral equations*, Dokl. Akad. Nauk SSSR **133** (1960), 1279–1282 = Soviet Math. Dokl. **1** (1960), 960–963. MR **23** #A2015.

7. E. A. Herman, *The symbol of the algebra of singular integral operators*, J. Math. Mech. **15** (1966), 147–155. MR **32** #4559.

8. L. Hörmander, *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517. MR **31** #4970.

9. S. M. Newberger, *The σ-symbol of the singular integral operators of Calderón and Zygmund*, Illinois J. Math. **9** (1965), 428–443. MR **31** #1579.

10. R. T. Seeley, *Integro-differential operators on vector bundles*, Trans. Amer. Math. Soc. **117** (1965), 167–204. MR **30** #3388.

University of Kansas, Lawrence, Kansas 66044