NORMAL CONTROL PROBLEMS HAVE NO MINIMIZING STRICTLY ORIGINAL SOLUTIONS

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ABSTRACT. We prove for a general optimal control problem that, in the absence of abnormal admissible extremals (solutions of a generalized Weierstrass E-condition), any control which is optimal in the set of original (ordinary) controls must also be optimal in the larger set of relaxed (measure-valued) controls.

1. We consider the model of an optimal control problem studied in [2]. This model was found applicable, among others, to unilateral control problems defined by ordinary differential and multidimensional integral equations [3], evasion problems [4], and conflicting control problems [5]. For the sake of completeness, we begin by restating the definition of this model. Let $T$ and $R$ be compact metric spaces and $\mu$ a positive and nonatomic Radon measure on $T$. We denote by $rpm(R)$ the set of regular Borel probability measures on $R$ endowed with the relative weak star topology of $C(R)^*$, by $@$ the class of $\mu$-measurable functions on $T$ to $R$ (original control functions), and by $S$ the set of $\mu$-measurable functions on $T$ to $rpm(R)$ (relaxed control functions). We embed $R$ in $rpm(R)$ and $@$ in $S$ by identifying each $r \in R$ with the Dirac measure at $r$. In turn, we view $S$ as a subset of $L^1(T, C(R))^*$, and endow it with the relative weak star topology, by identifying each $\sigma \in S$ with the functional $\phi \mapsto \int \mu(dt) \int \phi(t)(r) \sigma(t)(dr)$.

Now let $R$ be the real line, $\mathcal{X}$ a real topological vector space, $C$ a convex body in $\mathcal{X}$, $B$ a convex subset of a vector space (the set of control parameters), $m$ a positive integer, $x = (x_0, x_1, x_2) : \mathbb{S} \times B \rightarrow R \times R^m \times \mathcal{X}$ a given function, and

$$\mathcal{A}(\mathcal{U}) = \{ (\sigma, b) \in \mathcal{U} \times B \mid x_1(\sigma, b) = 0, x_2(\sigma, b) \in C \} \quad (\mathcal{U} \subset \mathcal{S}).$$

We say that $(\sigma, b)$ is a minimizing original (respectively relaxed) solution if it yields a minimum of $x_0$ on $\mathcal{A}(\mathcal{U})$ (respectively on $\mathcal{A}(S)$). A minimizing original solution is a minimizing strictly original solution if it is not at the same time a minimizing relaxed solution. We set $Q = \mathbb{S} \times B$, denote by $3_{m+1}$ the simplex $\{ (\theta^0, \cdots, \theta^m) \in R^{m+1} \mid \theta^j \geq 0, \}$
\[ \sum_{j=0}^{m} \theta^j \leq 1 \] and by \( D^x(q; q - q) \) the directional derivative \( \lim_{h \to 0} \alpha^{-1} [x(q + \alpha(q - q)) - x(q)] \). For \( q, q_0, q_1, \ldots, q_m \in Q \), we say that the function

\[ \theta \mapsto x \left( q + \sum_{j=0}^{m} \theta^j (q_j - q) \right) : S_m+1 \to \mathbb{R} \times \mathbb{R}^m \times \mathcal{X} \]

is differentiable at 0 if it has a Fréchet derivative at 0 relative to \( S_{m+1} \), i.e. if

\[ \lim_{\theta \to 0} \left| \theta \right|^{-1} \left[ x \left( q + \sum_{j=0}^{m} \theta^j (q_j - q) \right) - x(q) - \sum_{j=0}^{m} \theta^j D^x(q; q_j - q) \right] = 0 \]

in \( \mathbb{R} \times \mathbb{R}^m \times \mathcal{X} \) as \( |\theta| \to 0, \theta \in S_{m+1} \).

Points \( \bar{q} = (\bar{\sigma}, \bar{b}) \in S \times B \) and \( l = (l_0, l_1, l_2) \in [0, \infty) \times \mathbb{R}^m \times \mathbb{X}^* \) define an extremal \( (\bar{q}, l) \), and \( \bar{q} \) is extremal if \( \bar{q} \) and \( l \) satisfy the generalized Weierstrass E-condition (maximum principle)

\[ l \neq 0, \quad l(D^x(q; q - \bar{q})) \geq 0 \quad (q \in Q) \quad \text{and} \quad l_2(x_2(\bar{q})) \geq l(c) \quad (c \in C). \]

An extremal \( (\bar{q}, l) \) is admissible if \( \bar{q} = (\bar{\sigma}, \bar{b}) \in \alpha(\mathbb{S}) \); an extremal \( (\bar{q}, l) = (\bar{q}, l_0, l_1, l_2) \) is abnormal if \( l_0 = 0 \). The optimal control problem is normal if there exist no abnormal admissible extremals.

**Theorem I.** Assume that, for each choice of \( q, q_0, \ldots, q_m \in Q \), with \( q = (\sigma, \alpha) \) and \( q_i = (\sigma_i, \beta_i) \) \( (i = 0, 1, \ldots, m) \), the function

\[ (\sigma, \theta) \mapsto x \left( \sigma, b + \sum_{j=0}^{m} \theta^j (b_j - b) \right) : S \times S_{m+1} \to \mathbb{R} \times \mathbb{R}^m \times \mathcal{X} \]

is continuous and the function

\[ \theta \mapsto x \left( q + \sum_{j=0}^{m} \theta^j (q_j - q) \right) : S_{m+1} \to \mathbb{R} \times \mathbb{R}^m \times \mathcal{X} \]

is differentiable at 0. If \( (\bar{\sigma}, \bar{b}) \) is a minimizing strictly original solution then there exists an abnormal admissible extremal \( (\sigma^\#, b^\#, 0, l_1, l_2) \) such that \( x_0(\sigma^\#, b^\#) < x_0(\bar{\sigma}, \bar{b}) \).

**Proof.** Let \( (\bar{\sigma}, \bar{b}) \) be a minimizing strictly original solution. We set

\[ B' = B \times \mathbb{R}, \quad \mathcal{X}' = \mathcal{X} \times \mathbb{R}, \quad C' = C \times (-\infty, 0), \]

\[ x_2^0(\sigma, b') = \alpha, \quad x_1' (\sigma, b') = x_1(\sigma, b), \]

\[ x_2^0(\sigma, b') = (x_2(\sigma, b), x_0(\sigma, b) - x_0(\bar{\sigma}, \bar{b})) \quad (\sigma \in S, b' = (b, \alpha) \in B'), \]

\[ x' = (x_0', x_1', x_2'). \]

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We denote by $P$ the optimal control problem we are considering and by $P'$ the problem obtained by replacing $B$, $C$, $x$ with $B'$, $C'$ and $x'$, respectively. Since $(\bar{\sigma}, \bar{b})$ does not minimize $x_0$ on $\mathcal{A}(\mathcal{S})$, there exists $(\sigma^f, b^f) \in \mathcal{A}(\mathcal{S})$ such that $x_0(\sigma^f, b^f) < x_0(\bar{\sigma}, \bar{b})$. It follows that $x_1'(\sigma^f, b^f, 0) = 0$ and $x_2'(\sigma^f, b^f, 0) \in C'$. The argument of [2, 4.1, Proof of Theorem 2.2, p. 369], when applied to $P'$ and $(\sigma^f, b^f, 0)$, shows that either (a) there exists $l = (l_0, l_1, l_2') \in [0, \infty) \times \mathbb{R}^m \times (\mathcal{X} \times \mathbb{R})^a$ such that $(\sigma^f, b^f, 0, l_0, l_1, l_2')$ is an admissible extremal of $P'$, or (b) there exists $(\rho_1, b_1, \alpha_1) \in \mathcal{A} \times B'$ such that $x_1'(\rho_1, b_1, \alpha_1) = x_1(\rho_1, b_1) = 0$ and $x_2'(\rho_1, b_1, \alpha_1) \in C'$; hence $x_2(\rho_1, b_1) \in C$ and $x_0(\rho_1, b_1) < x_0(\bar{\sigma}, \bar{b})$. The alternative (b) must be discarded because it conflicts with the assumption that $(\bar{\sigma}, \bar{b})$ is a minimizing original solution. We set, in (a), $l_2' = (l_2, \lambda_0) \in \mathcal{X}^* \times \mathcal{R}$ and $q^f = (\sigma^f, b^f)$, and conclude that

$$l \neq 0, \quad l_0 \alpha + l_1 D x_1(q^f; q - q^f) + l_2(D x_2(q^f; q - q^f)) + \lambda_0 D x_0(q^f; q - q^f) \geq 0$$

$$\text{for } (\alpha \in \mathcal{R}, q = (\sigma, b) \in \mathcal{S} \times B)$$

and

$$l_2(x_2(q^f)) + \lambda_0 [x_0(q^f) - x_0(\bar{\sigma}, \bar{b})] \geq l_2(c) + \lambda \alpha \quad (c \in C, \alpha \in (-\infty, 0)).$$

Since $x_0(q^f) < x_0(\bar{\sigma}, \bar{b})$, these relations imply that $\lambda_0 = l_0 = 0$ and show that $(\sigma^f, b^f, 0, l_0, l_1)$ is an abnormal admissible extremal of $P$. Q.E.D.

2. Theorem I can be applied, under certain conditions, to problems where the original control functions are not a priori restricted to a compact set (e.g. to a problem of Bolza when its admissible extremals have uniformly bounded derivatives). Examples can be given [6, p. 118] of simple problems that possess minimizing strictly original solutions but, in view of Theorem I, these problems cannot be normal. If we add (to those of Theorem I) the assumptions that $\mathcal{A}(\mathcal{S})$ is nonempty and there exists a sequentially compact topology of $B$ such that $x$ is continuous on $\mathcal{S} \times B$ (or an appropriate subset) then, by [2, Theorems 2.1 and 2.2, pp. 362–363], there exists a minimizing relaxed solution and it is extremal. Thus, in normal problems, a minimizing original solution exists if and only if there exists an extremal point $(\bar{\sigma}, \bar{b}) \in \mathcal{A}(\mathcal{S})$ that minimizes $x_0$ among all extremal $(\sigma, b) \in \mathcal{A}(\mathcal{S})$. This suggests that the most promising approach to a theory of minimizing original solutions will remain the one that led to the justification of the Dirichlet principle and that McShane [1] applied in 1940 to the Bolza problem (using Young’s [7], [8] generalized curves as tools); namely, the investigation of conditions insuring that weak solutions of the problem (such as minimizing gen-
eralized curves or minimizing relaxed solutions) are also "classical" solutions.

We expect to publish elsewhere extensions of Theorem I with somewhat weaker hypotheses and with original control functions restricted by the condition \( p(t) \subseteq R^f(t) \) \( \mu \)-a.e., where \( R_f(\cdot) \) is a given \( \mu \)-measurable set-valued mapping. We shall also demonstrate the applicability of the model to functional-integral equations in \( C(T, \mathbb{R}^n) \) and \( L^p(T, \mathbb{R}^n) \).

References


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