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MANIFOLDS OF PIECEWISE LINEAR MAPS AND A RELATED NORMED LINEAR SPACE[1]

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1. **Spaces of piecewise linear maps.** Let X and Y be separable polyhedra, X compact and Y locally compact; for the moment let them be connected and of dimension >0. Denoting the separable hilbert space of square-summable sequences by l_2, a space is an l_2-*manifold* if separable, metrizable and locally homeomorphic to l_2. In [4] the author showed that the space $C(X, Y)$ of all continuous maps from X to Y with compact-open topology is an l_2-manifold. It is natural to ask whether the dense subspace $\mathrm{PL}(X, Y)$ consisting of all piecewise linear (p.l.) maps lies inside $C(X, Y)$ in some "nice" way. For example, is $\mathrm{PL}(X, Y)$ an infinite-dimensional manifold? and if so what is its model? and how are $\mathrm{PL}(X, Y)$ and $C(X, Y)$ related as manifolds?

To answer, let l_2^f be the (dense, incomplete) linear subspace of l_2 consisting of those sequences having only finitely many nonzero entries. Then we claim that $\mathrm{PL}(X, Y)$ is an l_2^f-manifold. A pair (M, N) is an (l_2, l_2^f)-*manifold pair*, if M is an l_2-manifold for which there is an open cover $\mathfrak{U}$ and open embeddings $\{f_U: U\to l_2 \mid U\in\mathfrak{U}\}$ such that for each $U\in\mathfrak{U}$, $f_U(U\cap N)=f_U(U)\cap l_2^f$. We claim that the pair $(C(X, Y), \mathrm{PL}(X, Y))$ is an (l_2, l_2^f)-manifold pair. Among other things, it follows that $\mathrm{PL}(X, Y)$ has a (metric) triangulation, and that if $\mathrm{PL}(X, Y)$ is contractible, then it is homeomorphic to l_2^f.

2. **Application: a normed linear space.** Before giving more details, we give a simple application. Consider $\mathrm{PL}(I, \boldsymbol{R})$, the space of p.l. paths in the real line, with the usual vector space structure. We have claimed that $\mathrm{PL}(I, \boldsymbol{R})$ is homeomorphic to l_2^f; but whereas the linear dimension of l_2^f is $\aleph_0$, the linear dimension of $\mathrm{PL}(I, \boldsymbol{R})$ is c. (Proof:

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[2] A special case of the theorem announced here is in the author's Ph.D. thesis written under Professor David W. Henderson at Cornell University. Supported by an I.B.M. Graduate Fellowship.

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suppose B is a linear basis of cardinality $\aleph < c$; then for each $g \in B$, let A_g be the finite set of vertices of the minimal subdivision of I on which g is linear; $\bigcup \{A_g | g \in B\}$ has cardinality $\aleph$, so there exists $x \in I$ which lies in no A_g; any p.l. map which is not linear on a neighborhood of x cannot be a finite linear combination of elements of B; this contradicts the fact that B is a linear basis.) Summarizing we have

EXAMPLE. *The normed linear space* $\mathrm{PL}(I, \boldsymbol{R})$ *is homeomorphic to* l_2^f *but has a different linear dimension.*

We mention this Example because we are told it is the first of its kind. Until recently it had been conjectured that any σ-compact normed linear space is topologically and linearly equivalent either to l_2^f or to the subspace of l_2 consisting of sequences $(x_1, x_2, \cdots)$ such that $0 \leqq x_i \leqq 1/2^i$ for all but a finite number of entries x_i. Spurred on by the above counterexample, Henderson and Pełczyński [5] have discovered an uncountable collection of σ-compact prehilbert spaces, no two of which are homeomorphic.

3. **The background.** By a topological characterization of l_2^f in l_2 we mean a set of topological conditions such that if M is homeomorphic to l_2 while $N \subset M$ satisfies the conditions, then (M, N) is homeomorphic (as a pair) to (l_2, l_2^f). Independently, Anderson in [1] and Bessaga-Pełczyński in [2] gave such a topological characterization. The work of Torunczyk in [7] and [8] is closely related. Developing their ideas, West gave the following topological characterization of (l_2, l_2)-manifold pairs:

THEOREM 1 (SEE THEOREM 6 OF [9]). *Let* (M, d) *be a metric space and let* $N \subset M$. *The pair* (M, N) *is an* (l_2, l_2^f)*-manifold pair if and only if*

(i) M *is an* l_2*-manifold,*

(ii) N *is the countable union of finite-dimensional compacta,*

(iii) *given* $\epsilon > 0$, *a pair* (A, B) *of finite-dimensional compacta and an embedding* $e: A \to M$ *such that* $e(B) \subset N$, *there exists an embedding* $e': A \to M$ *such that* $e'(A) \subset N$, $e' | B = e | B$ *and* $d(e(x), e'(x)) < \epsilon$ *for all* $x \in A$.

We should mention here that l_2^f-manifolds and (l_2, l_2^f)-manifold pairs have been studied by Henderson-West (an extensive list of properties appears in [6]) and by Chapman [3].

4. **The Main Theorem.** Throughout, we use the notation $C((X, X'), (Y, Y'))$ for the space of continuous functions from the pair (X, X') to the pair (Y, Y'), but if X' is empty we abbreviate to $C(X, Y)$. In the case of polyhedral pairs, we denote the corresponding spaces of p.l. maps by $\mathrm{PL}((X, X'), (Y, Y'))$ and $\mathrm{PL}(X, Y)$.

Before stating the Main Theorem we recall the following special case of Theorem (4.7) of [4]:

THEOREM 2. *Let K and L be simplicial complexes with K finite and L locally finite; let $|K|$ have dimension >0, let each component of $|L|$ have dimension >0, let K_0 be a subcomplex of K, and let $p\in|L|$. Then $C((|K|, |K_0|), (|L|, \{p\}))$ is an l_2-manifold. In particular, when K_0 is empty, $C(|K|, |L|)$ is an l_2-manifold.*

In general, we do not know whether Theorem 2 holds when $\{p\}$ is replaced by a more general subpolyhedron. But there is one well-known case given by Eells in [10] where it does hold. As in Theorem 2, we only state the polyhedral case:

THEOREM 3 (EELLS). *Let K and K_0 be as in Theorem 2; let L triangulate a C^2-manifold of dimension >0; let L_0 be a subcomplex which triangulates a C^2-submanifold of $|L|$. Then $C((|K|, |K_0|), (|L|, |L_0|))$ is an l_2-manifold.*

(*Note.* This way of stating Theorem 3 is valid because all separable Banach spaces are homeomorphic to l_2.)

THE MAIN THEOREM. *Let (K, K_0) and (L, L_0) be pairs of simplicial complexes, where K is finite and L is locally finite. Let*

$$M = C((|K|, |K_0|), (|L|, |L_0|)) \quad \text{and}$$

$$N = \mathrm{PL}((|K|, |K_0|), (|L|, |L_0|)).$$

Then (M, N) is an (l_2, l_2^f)-manifold pair whenever M is an l_2-manifold.

OUTLINE OF THE PROOF. We remark that in view of Theorems 2 and 3, M is nearly always an l_2-manifold, so the theorem is widely applicable. We must check that the conditions of Theorem 1 hold for the pair (M, N). Condition (i) holds by hypothesis. Condition (ii) is easily reduced to proving that, for each $k>0$, $\mathrm{PL}(\Delta^k, \mathbf{R})$ is the countable union of finite-dimensional compacta, where Δ^k is a euclidean k-simplex. This is not hard when $k=1$; and for any $k>0$ it is not hard to prove σ-compactness. The difficulty for $k>1$, and it is substantial, arises from the "finite-dimensional" requirement. We omit further details here. For condition (iii), the finite-dimensionality of A and the fact that M is an ANR are used to extend the embedding e to a map $f:D\to M$ where D is a compact polyhedron. Let $B'=f^{-1}(B)$. By simplicial approximation of a special sort, replace f by a nearby map $g:D\to M$ such that $g|B'=f|B'$ and $g(D-B')\cap g(B')=\emptyset$. It is also required of g that for each $x\in D-B'$, $g(x)$ is a p.l. map in N which is nonconstant on some component of $|K|$ and is linear on the simplexes of a barycentric subdivision of $|K|$ (the mesh of the subdivision depending on the distance of x from B'). Then for each $x \in D - B'$, a continuously varying p.l. homeomorphism $\psi(x)$ of $(|K|, |K_0|)$ is defined which is not the identity on any principal simplex of the particular barycentric subdivision of $|K|$ associated with the point x. This last condition ensures that $h(x) = g(x) \circ \psi(x)$ is different from $g(x)$. In fact, in this way, a map $h: D \to M$ is constructed so that h is near g, $h|B' = g|B' = f|B'$, $h(D - B') \cap h(B') = \emptyset$, $h(D - B') \subset N$ and $h|(D - B')$ is injective. Once such a map h has been constructed, it is clear that $e' = h|A$ is the required embedding.

References

1. R. D. Anderson, *On sigma-compact subsets of infinite-dimensional spaces*, Trans. Amer. Math. Soc. (to appear).

2. C. Bessaga and A. Pełczyński, *Estimated extension theorem, homogeneous collections and skeletons, and their applications to topological classification of linear metric spaces*, Fund. Math. **69** (1970), 153–190.

3. T. A. Chapman, *Dense sigma-compact subsets of infinite-dimensional manifolds*, Trans. Amer. Math. Soc. **154** (1971), 399–426.

4. R. Geoghegan, *On spaces of homeomorphisms, embeddings and functions*. I (to appear).

5. D. W. Henderson and A. Pełczyński, (to appear).

6. D. W. Henderson and J. E. West, *Triangulated infinite-dimensional manifolds*, Bull. Amer. Math. Soc. **76** (1970), 655–660.

7. H. Torunczyk, *On estimated extensions of homeomorphisms in metric spaces*, Bull. Acad. Polon. Sci. **18** (1970), 13–16.

8. ———, *Skeletonized sets in complete metric spaces and homeomorphisms of the Hilbert cube*, Bull. Acad. Polon. Sci. **18** (1970), 119–126.

9. J. E. West, *The ambient homeomorphy of an incomplete subspace of infinite-dimensional Hilbert spaces*, Pacific J. Math. **34** (1970), 257–268.

10. J. Eells, Jr., *On the geometry of function spaces*, Internat. Sympos. on Algebraic Topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 303–308. MR **20** #4878.

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