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PARTIAL DIFFERENTIAL EQUATIONS IN FISCHER-FOCK SPACES FOR THE HILBERT-SCHMIDT HOLOMORPHY TYPE

BY THOMAS A. W. DWYER, III[1]

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1. **Introduction.** Current work on the extension of function theory to infinite-dimensional domains has led to the consideration of classes of analytic functions defined on Banach spaces, with Fréchet derivatives of a given type, e.g., nuclear, compact or integral. The existence theory of partial differential equations in this setting follows from [G] for the nuclear bounded case, and is given in [D] for formal power series of α-β-γ-type. In this note we describe the duality theory (Theorem 1) and the existence theory (Theorem 2) of partial differential equations for a class of spaces of entire functions defined on a Hilbert space, with Fréchet derivatives given by Hilbert-Schmidt operators. When the underlying Hilbert space is finite-dimensional, we recover results in [T, Chapter 9], in [B] and in [NS] (Fischer space). When the underlying space is a Hilbert space of square-integrable functions, we obtain the wave functionals in the Fock representation of quantum field theory (cf. [NT]), subsuming some of the results proved independently in [R].

2. **Hilbert-Schmidt polynomials.** Let E be a Hilbert space over the complex field $\mathbf{C}$, with inner product $(\ |\ )$, and E' the dual of E, with the inner product $(u'|v')=(v|u)$ for $u'=(\ |u)$, $v'=(\ |v)$. Let $E'^{\vee n}=E'\vee\cdots\vee E'$ denote the n-fold symmetric product of E' [Gr, p. 191]. The Hilbert-Schmidt inner product on $E'^{\vee n}$ is characterized for decomposable elements by

$$(u_1'\vee\cdots\vee u_n' \mid v_1'\vee\cdots\vee v_n') = \frac{1}{n!}\sum_{\pi}(u_{\pi 1}'\mid v_1')\cdots(u_{\pi n}'\mid v_n'),$$

the summation extended over all permutations π of the indices. $E_H'^{\vee n}$ denotes the n-fold symmetric product equipped with the Hilbert-

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Schmidt inner product, and $(E_H'^{\vee n})^\wedge$ its completion.

For $n=1, 2, \cdots$, let $\mathcal{P}(^nE)$ denote the Banach space of continuous n-homogeneous polynomials P (obtained from continuous symmetric n-linear forms $A:E\times\cdots\times E\to\mathbf{C}$ by $P(x)=A(x,\cdots,x)$), with the supremum norm on the unit ball of E, and let $\mathcal{P}(^0E)=\mathbf{C}$ [N, p. 7].

PROPOSITION 1. *The formula* $i(u_1'\vee\cdots\vee u_n')=u_1'\cdots u_n'$, *where* $u_1'\cdots u_n'(x)=u_1'(x)\cdots u_n'(x)$ *for* $x\in E$ (*also* $u'^n=u'\cdots u'$), *defines a linear injection from* $E_H'^{\vee n}$ *into* $\mathcal{P}(^nE)$. *The continuous linear extension* $\tilde{\imath}$ *of* i *to* $(E_H'^{\vee n})^\wedge$ *is still injective. The image of* $\tilde{\imath}$, *denoted by* $\mathcal{P}_H(^nE)$, *is the Hilbert space of n-homogeneous Hilbert-Schmidt polynomials on* E, *with the inner product inherited from* $E_H'^{\vee n}$ *denoted by* $(\ |\ )_H$ *and the associated norm by* $\|\ \|_H$. *Let* $\mathcal{P}_H(^nE)'$ *be equipped with the dual inner product. Given* $P_n\in\mathcal{P}_H(^nE)$, *the formula* $P_n'(x')=(x'^n|P_n)_H$, *where* $x\in E$ *and* $x'^n=(\ |x)'$, *defines* $P_n'\in\mathcal{P}_H(^nE')$, *and the map* $(\ |P_n)_H\mapsto P_n'$ *is a Hilbert space isomorphism.*

3. Entire functions of Hilbert-Schmidt type.

PROPOSITION 2. *Given* $\rho>0$, *if* $f_n\in\mathcal{P}_H(^nE)$ *for each* n *and* $\sum_{n=0}^\infty\rho^n\|f_n\|_H^2/n!<\infty$ *then* $f=\sum_{n=0}^\infty f_n/n!$ *is an entire function of bounded type, i.e., f takes bounded sets into bounded sets. If* $\hat{d}^nf(x)$ *denotes the nth derivative polynomial of f at x then* $\hat{d}^nf(0)=f_n$. *The class of such functions, denoted by* $\mathfrak{F}_\rho(E)$, *is a Hilbert space, with the inner product* $(\ |\ )_\rho$ *given by*

$$(f|g)_\rho=\sum_{n=0}^\infty\rho^n\frac{1}{n!}(\hat{d}^nf(0)\,|\,\hat{d}^ng(0))_H$$

and the associated norm denoted by $\|\ \|_\rho$. *Clearly* $\|\ \|_\rho\leqq\|\ \|_\sigma$ *when* $\rho\leqq\sigma$. *Hence* $\mathfrak{F}_\infty(E)=\bigcap_{0<\rho<\infty}\mathfrak{F}_\rho(E)$ *with the projective limit topology is a countably Hilbert space, thus a reflexive Fréchet space, and* $\mathfrak{F}_0(E)=\bigcup_{0<\rho<\infty}\mathfrak{F}_\rho(E)$ *with the locally convex inductive limit topology is a bornological (DF)-space.*

THEOREM 1. *Let* $0\leqq\rho\leqq\infty$, *with* $\rho^{-1}=0$ *or* ∞ *when* $\rho=\infty$ *or* 0. *If* $f=\sum_{n=0}^\infty f_n/n!\in\mathfrak{F}_\rho(E)$ *and* $g'=\sum_{n=0}^\infty g_n'/n!\in\mathfrak{F}_{\rho^{-1}}(E')$, *and if* $g_n'\in\mathcal{P}_H(^nE')$ *corresponds to* $(\ |g_n)_H\in\mathcal{P}_H(^nE)'$, *then the series*

$$\langle f,g'\rangle=\sum_{n=0}^\infty\frac{1}{n!}(f_n\,|\,g_n)_H$$

defines a bilinear form, placing $\mathfrak{F}_\rho(E)$ *and* $\mathfrak{F}_{\rho^{-1}}(E')$ *in separating duality. The map* $g'\mapsto\langle\ ,g'\rangle$ *is a Hilbert space isomorphism (resp. a topological vector space isomorphism) from* $\mathfrak{F}_\rho(E)'$ *onto* $\mathfrak{F}_{\rho^{-1}}(E')$ *when*

$0<\rho<\infty$ *(resp.* $\rho=0$ *or* ∞*), and is the inverse of the Fourier-Borel transformation* [D].

SKETCH OF PROOF. Let $T\in\mathfrak{F}_\rho(E)'$. Since $\mathcal{P}_H({}^nE)$ is continuously imbedded in each $\mathfrak{F}_\rho(E)$, the restriction of T to $\mathcal{P}_H({}^nE)$ belongs to $\mathcal{P}_H({}^nE)'$, corresponding to $T_n'\in\mathcal{P}_H(E')$ given by $T_n'(x')=T(x'^n)$. The formula $\hat{T}(x')=T(\exp\circ x')$ defines the Fourier-Borel transform $\hat{T}:E'\to\mathbf{C}$ of T. We have in fact $\hat{T}=\sum_{n=0}^\infty T_n'/n!\in\mathfrak{F}_{\rho^{-1}}(E')$. Conversely, given $g'\in\mathfrak{F}_{\rho^{-1}}(E')$ each $\hat{d}^ng'(0)\in\mathcal{P}_H({}^nE')$ corresponds to a unique $(\ |g_n)_H\in\mathcal{P}_H({}^nE)'$. The formula

$$T_{g'}(f)=\sum_{n=0}^\infty\frac{1}{n!}(\hat{d}^nf(0)\mid g_n)_H$$

for $f\in\mathfrak{F}_\rho(E)$ defines $T_{g'}\in\mathfrak{F}_\rho(E)'$, and we get $\hat{T}_{g'}=g'$. This establishes the isomorphism of vector spaces and the duality $\langle f,g'\rangle=T_{g'}(f)$. Moreover, $\|g'\|_{\rho^{-1}}=\|T_{g'}\|$ (dual norm in $\mathfrak{F}_\rho(E)'$) when $0<\rho<\infty$. The continuity of the mappings from $\mathfrak{F}_\infty(E)'$ (resp. $\mathfrak{F}_0(E)'$) onto $\mathfrak{F}_0(E')$ (resp. $\mathfrak{F}_\infty(E')$) follows from the isometry from $\mathfrak{F}_\rho(E)'$ onto $\mathfrak{F}_{\rho^{-1}}(E')$ for $0<\rho<\infty$ by the properties of countably Hilbert spaces, bornological (DF)-spaces and their duals.

COROLLARY 1. *$\mathfrak{F}_0(E)$ is reflexive and complete. $\mathfrak{F}_\infty(E)$ and $\mathfrak{F}_0(E)$ are Montel spaces, in fact nuclear, if and only if E is finite-dimensional.*

SKETCH OF PROOF. The reflexivity, hence completeness, of the (DF)-space $\mathfrak{F}_0(E)$ follows from the duality. In the finite-dimensional case the nuclearity of $\mathfrak{F}_\infty(E)$, hence of $\mathfrak{F}_0(E)$, comes from the nuclearity of the injections $\mathfrak{F}_\sigma(E)\to\mathfrak{F}_\rho(E)$ for $\rho<\sigma$. E' is a closed barrelled subspace of $\mathfrak{F}_\infty(E)$ and of $\mathfrak{F}_0(E)$, so these spaces cannot be Montel or nuclear in the infinite-dimensional case.

4. **Partial differential operators of Hilbert-Schmidt type.** To define partial differential operators we need the following inequality.

PROPOSITION 3. *If $0\leqq k\leqq n$ and $P_n\in\mathcal{P}_H({}^nE)$ then $\hat{d}^kP_n(x)\in\mathcal{P}_H({}^kE)$, and for all $x\in E$ we have*:

$$\left\|\frac{1}{k!}\hat{d}^kP_n(x)\right\|_H\leqq\binom{n}{k}\|P_n\|_H\|x\|^{n-k}.$$

The proof, and others below, makes use of the following representation:

LEMMA 1. *Given an orthonormal basis $(e_i)_i$ of E, each $P_n\in\mathcal{P}_H({}^nE)$ is uniquely expressed as a limit in $\|\ \|_H$-norm by*

$$P_n = \sum_{i_1, \cdots, i_n} s_{i_1 \cdots i_n} e'_{i_1} \cdots e'_{i_n}$$

with symmetric coefficients $s_{i_1 \cdots i_n} \in \mathbf{C}$, *and*

$$\|P_n\|_H^2 = \sum_{i_1, \cdots, i_n} |s_{i_1 \cdots i_n}|^2.$$

We observe, however, that the $e'_{i_1} \cdots e'_{i_n}$ are not orthonormal.

By [N, §9, Lemma 1] we get $\hat{d}^n f(x) \in \mathcal{P}_H({}^nE)$ for all $x \in E$ when $\hat{d}^n f(0) \in \mathcal{P}_H({}^nE)$ and $\lim \sup_n \|\hat{d}^n f(0)/n!\|_H^{1/n} = 0$. We may then define: given $P = \sum_{n=0}^m P_n$ with $P_n \in \mathcal{P}_H({}^nE)$, the partial differential operator of Hilbert-Schmidt type $P(d)$ is given by

$$P(d)f(x) = \sum_{n=0}^{m} (\hat{d}^n f(x) \mid P_n)_H.$$

If $P = u'_1 \cdots u'_n$ then $P(d)$ is given by successive directional differentiation along $u_1, \cdots, u_n$. In particular, we are reduced to linear partial differential operators with constant coefficients in the finite-dimensional case. We also define the multiplication operator $P\cdot$ by $P\cdot f(x) = P(x)f(x)$.

Proposition 4. *If f is in $\mathfrak{F}_\sigma(E)$ then $P(d)f$ and $P\cdot f$ are in $\mathfrak{F}_\rho(E)$ for every $0<\rho<\sigma<\infty$. Hence if f is in $\mathfrak{F}_\infty(E)$ (resp. $\mathfrak{F}_0(E)$) then so are $P(d)f$ and $P\cdot f$.*

Easy counterexamples show that not all $f \in \mathfrak{F}_\rho(E)$ are mapped into $\mathfrak{F}_\rho(E)$ by $P(d)$ or $P\cdot$.

Theorem 2. *Let $0 \leqq \rho \leqq \infty$ and let $P(d)$ be any partial differential operator of Hilbert-Schmidt type: then for every $f \in \mathfrak{F}_\rho(E)$ there is some $g \in \mathfrak{F}_\rho(E)$ such that $P(d)g = f$.*

The proof uses the following lemmas:

Lemma 2. *If $P = \sum_{n=0}^m P_n$ and $P' = \sum_{n=0}^m P'_n$, where $(\ \mid P_n)_H \in \mathcal{P}_H({}^nE)'$ corresponds to $P'_n \in \mathcal{P}_H({}^nE')$ by $P'_n(x') = (x'^n \mid P_n)_H$, then $\langle P(d)f, g'\rangle = \langle f, P'\cdot g'\rangle$ for f and g' in the corresponding dual pairs (Theorem 1), finiteness and equality holding when either side is finite.*

The proof follows from a similar identity for the duality between $\mathcal{P}_H({}^nE)$ and $\mathcal{P}_H({}^nE')$, established first for polynomials of finite type (i.e., given by $E'^{\vee n}$), which are dense in $\mathcal{P}_H({}^nE)$.

Lemma 3. *If $0<\rho<\infty$, $f \in \mathfrak{F}_\rho(E)$ and $P = \sum_{n=0}^m P_n$ with $P_n \in \mathcal{P}_H({}^nE)$ then $\|P\cdot f\|_\rho \geqq \|P_m\|_\rho \|f\|_\rho$.*

The proof of the inequality uses a polynomial identity given in [T, Lemma 2.2] applied first to P of finite type and $f \in \bigcup_n \mathcal{P}_H({}^nE)$, extended by density to any Hilbert-Schmidt polynomial P, and finally to any $f \in \mathfrak{F}_\rho(E)$, with the help of the following facts: the pairs of operators $e'(d)$ and $e'\cdot$ obtained from an orthonormal basis $(e_i')_i$ of E' satisfy the correct commutation relations required over the polynomials; and Taylor series converge in $\| \ \|_\rho$-norm. The continuity of $P \cdot f \mapsto f$ follows from the inequality, and from the properties of projective and inductive limits in the cases $\rho = \infty$ and $\rho = 0$.

PROOF OF THEOREM 2. By Lemma 2 the transpose of $P(d)$ by $\langle \ , \ \rangle$ is $P'\cdot$, which has a continuous left inverse by Lemma 3 applied to $\mathfrak{F}_{\rho^{-1}}(E')$ (again $\rho^{-1} = 0$ when $\rho = \infty$). A standard Hahn-Banach argument gives the result.

PROPOSITION 5. *Let M be a measure space (e.g., locally compact), and make $E = L^2(M)$: then $P_n \in \mathcal{P}_H({}^nE)$ if and only if there is some $h_n \in L^2(M \times \cdots \times M)$, ($n$ variables and product measure), such that*

$$P_n(\alpha) = \int_M \cdots \int_M h_n(t_1, \cdots, t_n)\alpha(t_1) \cdots \alpha(t_n)\, dt_1 \cdots dt_n$$

for every $\alpha \in L^2(M)$. The function h_n can be uniquely chosen to be symmetric, and then $\|P_n\|_H = \|h_n\|_{L^2}$.

It follows that the functions $f \in \mathfrak{F}_\rho(E)$ are the Fock functionals of [NT] and [R], and the partial differentials $P_n(d)f(\alpha)$ are the functional derivatives $h_n f^{(n)}(\alpha)$ of [R], where h_n corresponds to P_n by the formula given above. The proof of Proposition 5 follows from the Hilbert space isomorphism between $(L^2(M)_H^{\vee n})^\wedge$ and symmetric $L^2(M \times \cdots \times M)$.

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NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115