PARTIAL DIFFERENTIAL EQUATIONS IN FISCHER-FOCK SPACES FOR THE HILBERT-SCHMIDT HOLOMORPHY TYPE

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1. Introduction. Current work on the extension of function theory to infinite-dimensional domains has led to the consideration of classes of analytic functions defined on Banach spaces, with Fréchet derivatives of a given type, e.g., nuclear, compact or integral. The existence theory of partial differential equations in this setting follows from [G] for the nuclear bounded case, and is given in [D] for formal power series of $\alpha\beta\gamma$-type. In this note we describe the duality theory (Theorem 1) and the existence theory (Theorem 2) of partial differential equations for a class of spaces of entire functions defined on a Hilbert space, with Fréchet derivatives given by Hilbert-Schmidt operators. When the underlying Hilbert space is finite-dimensional, we recover results in [T, Chapter 9], in [B] and in [NS] (Fischer space). When the underlying space is a Hilbert space of square-integrable functions, we obtain the wave functionals in the Fock representation of quantum field theory (cf. [NT]), subsuming some of the results proved independently in [R].

2. Hilbert-Schmidt polynomials. Let $E$ be a Hilbert space over the complex field $\mathbb{C}$, with inner product $\langle \cdot, \cdot \rangle$, and $E'$ the dual of $E$, with the inner product $\langle u', v' \rangle = \langle v, u \rangle$ for $u' = (u), v' = (v)$. Let $E_n^{\vee} = E'^* \cdots \vee E'^*$ denote the $n$-fold symmetric product of $E'$ [Gr, p. 191]. The Hilbert-Schmidt inner product on $E_n^{\vee}$ is characterized for decomposable elements by

$$
\langle u'_1 \vee \cdots \vee u'_n \mid v'_1 \vee \cdots \vee v'_n \rangle = \frac{1}{n!} \sum_{\pi} \langle u'_{\pi_1} \mid v'_{\pi_1} \rangle \cdots \langle u'_{\pi_n} \mid v'_{\pi_n} \rangle,
$$

the summation extended over all permutations $\pi$ of the indices. $E_n^{\vee}$ denotes the $n$-fold symmetric product equipped with the Hilbert-Schmidt inner product.
Schmidt inner product, and \((E^H)^n\) its completion.

For \(n = 1, 2, \cdots\), let \(\mathcal{C}(\mathcal{E})\) denote the Banach space of continuous \(n\)-homogeneous polynomials \(P\) (obtained from continuous symmetric \(n\)-linear forms \(A: \mathcal{E} \times \cdots \times \mathcal{E} \to \mathbb{C}\) by \(P(x) = A(x, \cdots, x)\)), with the supremum norm on the unit ball of \(E\), and let \(\mathcal{C}(\mathcal{E}) = \mathbb{C}\) [N, p. 7].

**Proposition 1.** The formula \(i\left(u_1, \cdots, u_n\right) = u_1 \cdots u_n\), where \(u_1 \cdots u_n(x) = u_1(x) \cdots u_n(x)\) for \(x \in E\) (also \(u^n = u' \cdots u'\)), defines a linear injection from \(E^H\) into \((E^H)^n\). The continuous linear extension \(\overline{i}\) of \(i\) to \((E^H)^n\) is still injective. The image of \(\overline{i}\), denoted by \(\mathcal{P}(E)^n\), is the Hilbert space of \(n\)-homogeneous Hilbert-Schmidt polynomials on \(E\), with the inner product inherited from \(E^H\) denoted by \(\langle \cdot, \cdot \rangle_H\) and the associated norm by \(\| \cdot \|_H\). Let \(\mathcal{P}(E)^n\) be equipped with the dual inner product. Given \(P_n \in \mathcal{P}(E)^n\), the formula \(P_n(x') = (x' \cdot P_n)_H\), where \(x \in E\) and \(x' = (x')'\), defines \(P_n' \in \mathcal{P}(E)^n\), and the map \(\mathcal{P}(E)^n \to P_n'\) is a Hilbert space isomorphism.

### 3. Entire functions of Hilbert-Schmidt type.

**Proposition 2.** Given \(\rho > 0\), if \(f_n \in \mathcal{P}(E)^n\) for each \(n\) and \(\sum_{n=0}^\infty \rho^n \|f_n\|_H^n/n! < \infty\) then \(f = \sum_{n=0}^\infty f_n/n!\) is an entire function of bounded type, i.e., \(f\) takes bounded sets into bounded sets. If \(d^nf(x)\) denotes the \(n\)th derivative polynomial of \(f\) at \(x\) then \(d^nf(0) = f_n\). The class of such functions, denoted by \(\mathcal{S}_\rho(E)\), is a Hilbert space, with the inner product \(\langle \cdot, \cdot \rangle_\rho\) given by

\[
(f|g)_\rho = \sum_{n=0}^\infty \frac{1}{n!} \langle d^nf(0)|d^ng(0)\rangle_H
\]

and the associated norm denoted by \(\| \cdot \|_\rho\). Clearly \(\| \cdot \|_\rho \leq \| \cdot \|_\sigma\) when \(\rho \leq \sigma\).

Hence \(\mathcal{S}_\rho(E) = \bigcap_{0 < \rho < \infty} \mathcal{S}_\rho(E)\) with the projective limit topology is a countably Hilbert space, thus a reflexive Fréchet space, and \(\mathcal{S}_0(E) = \bigcup_{0 < \rho < \infty} \mathcal{S}_\rho(E)\) with the locally convex inductive limit topology is a bornological \((DF)\)-space.

**Theorem 1.** Let \(0 \leq \rho \leq \infty\), with \(\rho^{-1} = 0\) or \(\infty\) when \(\rho = \infty\) or \(0\). If \(f = \sum_{n=0}^\infty f_n/n! \in \mathcal{S}_\rho(E)\) and \(g' = \sum_{n=0}^\infty g'_n/n! \in \mathcal{S}_\rho^{-1}(E')\), and if \(g'_n \in \mathcal{S}_H^n(E')\) corresponds to \(\langle \cdot, g'_n\rangle_H \in \mathcal{S}_H^n(E')\), then the series

\[
\langle f, g' \rangle = \sum_{n=0}^\infty \frac{1}{n!} \langle f_n, g'_n \rangle_H
\]

defines a bilinear form, placing \(\mathcal{S}_\rho(E)\) and \(\mathcal{S}_\rho^{-1}(E')\) in separating duality. The map \(g' \mapsto \langle \cdot, g' \rangle\) is a Hilbert space isomorphism (resp. a topological vector space isomorphism) from \(\mathcal{S}_\rho(E)\) onto \(\mathcal{S}_\rho^{-1}(E')\) when
0 < p < \infty \) (resp. \( p = 0 \) or \( \infty \)), and is the inverse of the Fourier-Borel transformation \([D]\).

**Sketch of Proof.** Let \( T \in \mathcal{F}_\rho(E)' \). Since \( \mathcal{F}_\rho(E) \) is continuously imbedded in each \( \mathcal{F}_\rho(E) \), the restriction of \( T \) to \( \mathcal{F}_\rho(E) \) belongs to \( \mathcal{F}_\rho(E)' \), corresponding to \( T'_n \in \mathcal{F}_\rho(E)' \) given by \( T'_n(x') = T(x^n) \). The formula \( \hat{T}(x') = T(\exp x') \) defines the Fourier-Borel transform \( \hat{T}: E' \to \mathbb{C} \) of \( T \). We have in fact \( \hat{T} = \sum_{n=0}^{\infty} T'_n/n! \in \mathcal{F}_\rho^{-1}(E') \). Conversely, given \( g' \in \mathcal{F}_\rho^{-1}(E') \) each \( \hat{d}^n g'(0) \in \mathcal{F}_\rho(E)' \) corresponds to a unique \( (g_n)_{n} \in \mathcal{F}_\rho(E)' \). The formula

\[
T'_n(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle d^n f(0) | g_n \rangle
\]

for \( f \in \mathcal{F}_\rho(E) \) defines \( T'_n \in \mathcal{F}_\rho(E)' \), and we get \( \hat{T}' = g' \). This establishes the isomorphism of vector spaces and the duality \( \langle f, g' \rangle = T(\hat{f}) \). Moreover, \( \|g'\|_{\rho^{-1}} = \|T'_n\|_{\rho^{-1}} \) (dual norm in \( \mathcal{F}_\rho(E)' \)) when \( 0 < \rho < \infty \).

The continuity of the mappings from \( \mathcal{F}_\rho(E)' \) (resp. \( \mathcal{F}_\rho(E)' \)) onto \( \mathcal{F}_\rho(E)' \) (resp. \( \mathcal{F}_\rho(E)' \)) follows from the isometry from \( \mathcal{F}_\rho(E)' \) onto \( \mathcal{F}_\rho^{-1}(E') \) for \( 0 < \rho < \infty \) by the properties of countably Hilbert spaces, bornological (DF)-spaces and their duals.

**Corollary 1.** \( \mathcal{F}_0(E) \) is reflexive and complete. \( \mathcal{F}_\infty(E) \) and \( \mathcal{F}_\sigma(E) \) are Montel spaces, in fact nuclear, if and only if \( E \) is finite-dimensional.

**Sketch of Proof.** The reflexivity, hence completeness, of the (DF)-space \( \mathcal{F}_0(E) \) follows from the duality. In the finite-dimensional case the nuclearity of \( \mathcal{F}_\infty(E) \), hence of \( \mathcal{F}_\sigma(E) \), comes from the nuclearity of the injections \( \mathcal{F}_{\sigma}(E) \to \mathcal{F}_\rho(E) \) for \( \rho < \sigma \). \( E' \) is a closed barrelled subspace of \( \mathcal{F}_{\sigma}(E) \) and of \( \mathcal{F}_\rho(E) \), so these spaces cannot be Montel or nuclear in the infinite-dimensional case.

4. Partial differential operators of Hilbert-Schmidt type. To define partial differential operators we need the following inequality.

**Proposition 3.** If \( 0 \leq k \leq n \) and \( P_n \in \mathcal{F}_\rho(E) \) then \( \hat{d}^k P_n(x) \in \mathcal{F}_\rho(E) \), and for all \( x \in E \) we have:

\[
\left\| \frac{1}{k!} \hat{d}^k P_n(x) \right\|_{H} \leq \binom{n}{k} \left\| P_n \right\|_H \|x\|^{n-k}.
\]

The proof, and others below, makes use of the following representation:

**Lemma 1.** Given an orthonormal basis \( (e_i)_i \) of \( E \), each \( P_n \in \mathcal{F}_\rho(E) \) is uniquely expressed as a limit in \( \| \cdot \|_{H} \)-norm by
\[ P_n = \sum_{i_1, \ldots, i_n} s_{i_1} \cdots s_{i_n} e_{i_1} \cdots e_{i_n} \]

with symmetric coefficients \( s_{i_1} \cdots s_{i_n} \in \mathbb{C} \), and

\[ \|P_n\|^2_H = \sum_{i_1, \ldots, i_n} |s_{i_1} \cdots s_{i_n}|^2. \]

We observe, however, that the \( e_{i_1} \cdots e_{i_n} \) are not orthonormal.

By [N, §9, Lemma 1] we get \( \hat{d}^n f(x) \in \mathscr{O}_H(\mathbb{E}) \) for all \( x \in \mathbb{E} \) when \( \hat{d}^n f(0) \in \mathscr{O}_H(n\mathbb{E}) \) and \( \limsup \|\hat{d}^n f(0)/n^n\|_H = 0 \). We may then define:

given \( P = \sum_{n=0}^m P_n \) with \( P_n \in \mathscr{O}_H(n\mathbb{E}) \), the partial differential operator of Hilbert-Schmidt type \( P(d) \) is given by

\[ P(d)f(x) = \sum_{n=0}^m (\hat{d}^n f(x) | P_n)_H. \]

If \( P = u_1' \cdots u_n' \) then \( P(d) \) is given by successive directional differentiation along \( u_1, \ldots, u_n \). In particular, we are reduced to linear partial differential operators with constant coefficients in the finite-dimensional case. We also define the multiplication operator \( P \cdot \) by \( P \cdot f(x) = P(x)f(x) \).

**Proposition 4.** If \( f \) is in \( \mathfrak{F}_\sigma(\mathbb{E}) \) then \( P(d)f \) and \( P \cdot f \) are in \( \mathfrak{F}_\sigma(\mathbb{E}) \) for every \( 0 < \rho < \sigma < \infty \). Hence if \( f \) is in \( \mathfrak{F}_\sigma(\mathbb{E}) \) (resp. \( \mathfrak{F}_\rho(\mathbb{E}) \)) then so are \( P(d)f \) and \( P \cdot f \).

Easy counterexamples show that not all \( f \in \mathfrak{F}_\rho(\mathbb{E}) \) are mapped into \( \mathfrak{F}_\rho(\mathbb{E}) \) by \( P(d) \) or \( P \cdot \).

**Theorem 2.** Let \( 0 \leq \rho \leq \infty \) and let \( P(d) \) be any partial differential operator of Hilbert-Schmidt type: then for every \( f \in \mathfrak{F}_\rho(\mathbb{E}) \) there is some \( g \in \mathfrak{F}_\rho(\mathbb{E}) \) such that \( P(d)g = f \).

The proof uses the following lemmas:

**Lemma 2.** If \( P = \sum_{n=0}^m P_n \) and \( P' = \sum_{n=0}^m P'_n \), where \( (P_n)_H \in \mathscr{O}_H(n\mathbb{E}) \) corresponds to \( P'_n \in \mathscr{O}_H(n\mathbb{E}) \) by \( P'_n(x') = (x'|P_n)_H \), then \( \langle P(d)f, g' \rangle = \langle f, P'(g') \rangle \) for \( f \) and \( g' \) in the corresponding dual pairs (Theorem 1), finiteness and equality holding when either side is finite.

The proof follows from a similar identity for the duality between \( \mathscr{O}_H(n\mathbb{E}) \) and \( \mathscr{O}_H(n\mathbb{E})' \), established first for polynomials of finite type (i.e., given by \( E' \wedge n \)), which are dense in \( \mathscr{O}_H(n\mathbb{E}) \).

**Lemma 3.** If \( 0 < \rho < \infty \), \( f \in \mathfrak{F}_\rho(\mathbb{E}) \) and \( P = \sum_{n=0}^m P_n \) with \( P_n \in \mathscr{O}_H(n\mathbb{E}) \) then \( \|P \cdot f\|_\rho \geq \|P_m\|_\rho \|f\|_\rho \).
The proof of the inequality uses a polynomial identity given in [T, Lemma 2.2] applied first to $P$ of finite type and $f \in \bigcup_n \mathcal{H}(\mathcal{E})$, extended by density to any Hilbert-Schmidt polynomial $P$, and finally to any $f \in \mathcal{H}(\mathcal{E})$ with the help of the following facts: the pairs of operators $e'(d)$ and $e'$ obtained from an orthonormal basis $(e_i')$ of $\mathcal{E}$ satisfy the correct commutation relations required over the polynomials; and Taylor series converge in $\| \cdot \|_p$-norm. The continuity of $P \cdot f \mapsto f$ follows from the inequality, and from the properties of projective and inductive limits in the cases $p = \infty$ and $p = 0$.

**Proof of Theorem 2.** By Lemma 2 the transpose of $P(d)$ by $\langle \cdot , \cdot \rangle$ is $P'$, which has a continuous left inverse by Lemma 3 applied to $\mathcal{F}(\mathcal{E})$ (again $p^{-1} = 0$ when $p = \infty$). A standard Hahn-Banach argument gives the result.

**Proposition 5.** Let $M$ be a measure space (e.g., locally compact), and make $E = L^2(M)$: then $P_n \in \mathcal{H}(\mathcal{E})$ if and only if there is some $h_n \in L^2(M \times \cdots \times M)$, $(n$ variables and product measure), such that

$$P_n(\alpha) = \int_M \cdots \int_M h_n(t_1, \cdots, t_n) \alpha(t_1) \cdots \alpha(t_n) \, dt_1 \cdots dt_n$$

for every $\alpha \in L^2(M)$. The function $h_n$ can be uniquely chosen to be symmetric, and then $\|P_n\|_p = \|h_n\|_L^2$.

It follows that the functions $f \in \mathcal{H}(\mathcal{E})$ are the Fock functionals of [NT] and [R], and the partial differentials $P_n(d)f(\alpha)$ are the functional derivatives $h_n f^{(n)}(\alpha)$ of [R], where $h_n$ corresponds to $P_n$ by the formula given above. The proof of Proposition 5 follows from the Hilbert space isomorphism between $(L^2(M)^n)^\wedge$ and symmetric $L^2(M \times \cdots \times M)$.

**References**


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