ON THE MEAN CURVATURE OF SUBMANIFOLDS OF EUCLIDEAN SPACE

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Let \( x : M^n \to E^m \) be an immersion of an \( n \)-dimensional manifold \( M^n \) in a euclidean space \( E^m \) of dimension \( m \) \((m > n > 1)\), and let \( \nabla \) and \( \nabla' \) be the covariant differentiations of \( M^n \) and \( E^m \), respectively. Let \( u \) and \( v \) be two tangent vector fields on \( M^n \). Then the second fundamental form \( h \) is given by

\[
\nabla'_u v = \nabla_u v + h(u, v).
\]

If \( \{e_1, \ldots, e_n\} \) is an orthonormal basis in the tangent space \( T_p(M) \) at \( p \in M^n \), then the mean curvature vector \( H(p) \) at \( p \) is given by

\[
H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i).
\]

Let \( \langle \ , \ \rangle \) denote the scalar product of \( E^m \). If there exists a function \( f \) on \( M \) such that \( \langle h(u, v), H \rangle = f(u, v) \) for all tangent vector fields \( u, v \) on \( M^n \), then \( M^n \) is called a pseudo-umbilical submanifold of \( E^m \).

If the covariant derivative of \( H \) in \( E^m \) is tangent to \( x(M^n) \) everywhere, then \( H \) is said to be parallel in the normal bundle. In [2], [3], the author proved that if \( M^n \) is closed, then the mean curvature vector \( H \) satisfies

\[
\int_{M^n} \langle H, H \rangle^{n/2} dV \geq c_n,
\]

where \( dV \) denotes the volume element of \( M^n \) and \( c_n \) is the area of the unit \( n \)-sphere. The equality sign of (3) holds when and only when \( M^n \) is imbedded as a hypersphere in an \((n + 1)\)-dimensional linear subspace of \( E^m \). It is interesting to know whether the inequality (3) can be improved for some special submanifolds of \( E^m \).

The main purpose of this paper is to announce some results in this direction together with some results on pseudo-umbilical submanifolds. Details will appear elsewhere.


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By studying the behaviors of the mean curvature vector $H$ we can prove

**Lemma 1.** The position vector field $X$ of $M^n$ in $E^m$ is parallel to the mean curvature vector $H$ when and only when $M^n$ is either a minimal submanifold of $E^m$ or a minimal submanifold of a hypersphere of $E^m$ centered at the origin.

By using Lemma 1, we can prove

**Proposition 1** (Yano-Chen [5]). $M^n$ is a pseudo-umbilical submanifold of $E^m$ such that the mean curvature vector $H$ is parallel in the normal bundle when and only when $M^n$ is either a minimal submanifold of $E^m$ or a minimal submanifold of a hypersphere of $E^m$.

If the codimension is equal to 2, then a pseudo-umbilical submanifold of $E^m$ with constant mean curvature is always a pseudo-umbilical submanifold such that the mean curvature vector is parallel in the normal bundle. Hence we have

**Proposition 2.** $M^n$ is a pseudo-umbilical submanifold of $E^{n+2}$ with constant mean curvature when and only when $M^n$ is either a minimal submanifold of $E^{n+2}$ or a minimal hypersurface of a hypersphere of $E^{n+2}$.

Let $F$ be a field and $H_i(M^n, F)$ denote the $i$th cohomology group of $M^n$ over the field $F$. Let $\beta(M^n) = \max \{ \sum_{i=0}^n \dim H_i(M^n, F); F \text{ fields} \}$. Then, by verifying the properties of the length of second fundamental form $h$, we can prove

**Theorem I.** Let $M^n$ be an $n$-dimensional closed manifold immersed in $E^m$ with nonnegative scalar curvature. Then we have

$$\int_{M^n} \langle H, H \rangle^{n/2} dV > a \beta(M^n),$$

where

$\begin{align*}
\text{if } n \text{ is even,} & \quad a = (4n^n)^{-1/2c_n}, \\
\text{if } n \text{ is odd,} & \quad a = (2n^n c_{m-n-1} c_{m+n-1})^{-1/2}(c_{2n})^{1/2} c_{m-1},
\end{align*}$

**Theorem II.** Let $M^2$ be a flat torus in $E^4$. Then we have

$$\int_{M^2} \langle H, H \rangle dV \geq 2\pi^2.$$

Then the equality sign of (6) holds when and only when $M^2$ is a Clifford torus in $E^4$.

**Proof (sketch).** The proof of (6) follows from a direct computation of the first curvature of second kind, $\lambda_1(\tilde{\phi})$, (for the definition,
see [1]) and the relations between the mean curvature and $\lambda_1$. If the equality of (6) holds, then we can prove that $M^2$ is a minimal surface of a 3-sphere in $E^4$. From this we see that $M^2$ is a Clifford torus in $E^4$. The converse of this is trivial.

For each unit normal vector $e$ to $x(M^n)$ at $x(p)$, let $h_e$ be the linear transformation from the tangent space $T_p(M)$ into itself defined by

$$(7) \quad \langle h_e(u), v \rangle = \langle h(u, v), e \rangle$$

for all tangent vectors $u, v$ at $p$. Let $K(p, e) = \det(h_e)$. Then $K(p, e)$ is called the Lipschitz-Killing curvature at $(p, e)$. By deriving some integral formulas for the $\alpha$th curvatures of first and second kinds (for the definitions, see [1]), we can prove

**Theorem III.** Let $M^2$ be an oriented closed surface in $E^m$. If $M^2$ is contained in a hypersphere of $E^m$, then $M^2$ is a pseudo-umbilical surface of $E^m$ when and only when the Lipschitz-Killing curvature in the unit direction of the mean curvature vector $H$ is maximal over the fibre of the unit normal bundle $B_v, B_v = \{ (p, e) : p \in M^2, e \text{ a unit normal vector in } E^m \text{ at } x(p) \}$.

**Theorem IV (added in proof).** The Veronese surface in $E^5$, the generalized Clifford tori in $E^{n+2}$ and the $n$-sphere in $E^{n+p}$ are the only closed pseudo-umbilical submanifolds $M^n$ of $E^{n+p}$ with mean curvature vector nowhere zero satisfying

$$(8) \quad R \geq \frac{n(p - 1)\langle H, H \rangle}{2p - 3} \left[ (n - 1) \left( \frac{2p - 3}{p - 1} \right) - 1 \right]$$

where $R$ denotes the scalar curvature of $M^n$.

The proof of this theorem will appear in a forthcoming paper “Pseudo-umbilical submanifolds in a Riemannian manifold of constant curvature. II”.

**References**

4. ———, On the total curvature of immersed manifolds. II. Mean curvature and length of second fundamental form (to appear).

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