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ON THE CONVERGENCE OF MULTIPLE FOURIER SERIES

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We continue from [2].

THEOREM. *Let P be an open polygonal region in R^2, containing the origin. Set $\lambda P = \{(\lambda x, \lambda y) \mid (x, y) \in P\}$ for $\lambda > 0$. Then for*

$$f \sim \sum_{m,n=-\infty}^{\infty} a_{mn} \exp[i(mx + ny)]$$

in $L^2([0, 2\pi] \times [0, 2\pi])$, we have

$$f(x, y) = \lim_{\lambda\to\infty} \sum_{(m,n)\in\lambda P} a_{mn} \exp[i(mx+ny)]$$

almost everywhere.

Surprisingly, this is an easy consequence of Carleson's theorem [1] on convergence of Fourier series of one variable.

PROOF. It is enough to prove the maximal inequality

$$\left\| \sup_\lambda \left| \sum_{(m,n)\in\lambda P} a_{mn} \exp[i(mx + ny)] \right| \right\|_2 \leqq C\|f\|_2. \tag{1}$$

Inequality (1) follows from the special case in which P is a triangle with a vertex at the origin; for any polygon breaks up into triangles, and the characteristic function of any triangle is a linear combination of characteristic functions of triangles with vertices at zero. Consequently, we can assume P has the form $P = \{(x, y) \in S \mid (x, y) \cdot t < a\}$, where S is a sector of angle $< \pi$ emanating from the origin, $t \in R^2$, and $a \in R^1$. Thus (1) is equivalent to

$$\left\| \sup_{b\in R^1} \left| \sum_{(m,n)\in S;\,(m,n)\cdot t<b} a_{mn} \exp[i(mx + ny)] \right| \right\|_2 \leqq C\|f\|_2. \tag{2}$$

Evidently, it suffices to prove (2) for rational t (with C independent of t), and to do so it is clearly enough to deal with the case $t = (p, q)$ where p and q are relatively prime integers. Finding integers r and s for which $pr - qs = 1$, we let the matrix $A = \binom{p\ s}{q\ r} \in SL(2, Z)$ act as an automorphism of the 2-torus. Under the action of A, (2) becomes

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$$(3) \qquad \left\| \sup_b \left| \sum_{(m',n')\in S';\, m'<b} a_{m'n'} \exp[i(m'x' + n'y')] \right| \right\|_2 \leqq C\|f'\|_2.$$

Here,

$$S' = A^{-1}(S), f'(x', y') = f(A(x', y')) \text{ and } \sum_{m',n'} a_{m'n'} \exp[i(m'x' + n'y')]$$

is the Fourier series of f'. Note that C is unchanged from (2) to (3). However, (3) follows at once by applying the Carleson-Hunt theorem of [3] to the function $g(\cdot, y')$ for each y', where $g'(x', y') \sim \sum_{(m',n')\in S'} a_{m'n'} \exp[i(m'x'+n'y')]$. Q.E.D.

REMARKS. 1. The same proof applies to all L^p, $p>1$, and also (with some padding) to polyhedra in n variables.

2. For P a rectangle, a more precise argument, discovered independently by P. Sjölin [4], proves convergence of double Fourier series under minimal growth conditions on f. The best known hypotheses are $f\in L(\log L)^2 \log\log L$ for P a rectangle, and $f\in L(\log L)^3 \log\log L$ in general. The relationship of our proof to Sjölin's is not clear.

3. N. Tevzadze [5] has shown that for $f\in L^2([0, 2\pi]\times[0, 2\pi])$ and for any monotone sequence of rectangles $R_1\subseteq R_2\subseteq R_3\subseteq \cdots$ in R^2 with sides parallel to the coordinate axes,

$$f(x, y) = \lim_{i\to\infty} \sum_{(m,n)\in R_i} a_{mn} \exp[i(mx + ny)]$$

almost everywhere.

Compare with the counterexamples of [2].

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