GEOMETRIC THEORY OF DIFFERENTIAL EQUATIONS.  
THE LJAPUNOV INTEGRAL FOR  
MONOTONE COEFFICIENTS  

BY H. GUGGENHEIMER

Communicated by James Serrin, March 3, 1971

An equation
\[(1) \quad x'' + p(t)x = 0, \quad -\infty < t < \infty, \quad p(t) > 0,\]
can be considered as the Frenet equation of a locally convex curve
\[x(t) = (x_1(t), x_2(t))\]
in unimodular centroaffine differential geometry [1]. The osculating ellipse \(E_u\) at \(x(u)\) is the solution of
\[y'' + p(u)y = 0, \quad y(u) = x(u), \quad y'(u) = x'(u).\]

We prove an unimodular centroaffine Kneser theorem:

**Theorem 1.** If \(p(t)\) is strictly monotone and differentiable in an interval \([a, b]\), then every osculating ellipse \(E_t\), \(t \in [a, b]\), contains all osculating ellipses of smaller area defined on the same interval in its interior.

The area of the osculating ellipse is proportional to \(p(t)^{-1/2}\). By the Jordan curve theorem, the assertion is true if it is true for neighboring points. Then it is easily checked that a pair of conjugate diameters of the smaller ellipse is in the interior of the larger one. The approximation and convergence theorems of convexity imply:

**Theorem 2.** If \(p(t)\) is monotone and continuous in \([a, b]\), then every osculating ellipse \(E_t\), \(t \in [a, b]\), contains all osculating ellipses of smaller area defined on the same interval.

The parameter \(t - u\) is equal to two times the area covered by the radius vector of \(x(t)\) and \(\int_u^t p(\tau) d\tau\) is equal to two times the area covered by the radius vector of the polar reciprocal \(x^*(t)\) of \(x\) for the unit circle, if \(x(t)\) is a curve of unit Wronskian [1, §3]. For \(p\) monotone increasing, the curve \(x\) and the osculating ellipses \(E_t\) \((t \geq u)\) are contained in \(E_u\), and \(x^*(\tau), E_r^*\) \((u \leq \tau \leq t)\) are in \(E_t^*\).

Let \(\phi(u)\) be the conjugate point of \(u\) for (1), i.e., the zero following...
of a nontrivial solution of (1) that vanishes at \( u \). Let \( \psi(u) \) be the co-conjugate point of \( u \) for (1), i.e., the zero following \( u \) of the derivative of a nontrivial solution of (1) whose derivative vanishes at \( u \). A major topic in the study of equations (1) are estimates of the Lyapunov integral

\[
L(u) = [\phi(u) - u] \int_u^{\phi(u)} p(t) \, dt.
\]

We suppose that \( p(t) \) is monotone increasing and continuous and that \( \phi(u) < \infty \). As an application of Theorem 2, we have

\[
\pi p(\phi(u))^{-1/2} \leq \phi(u) - u \leq \pi p(\phi(u))^{-1/2},
\]

\[
\pi p(u)^{1/2} \leq \int_u^{\phi(u)} p(t) \, dt.
\]

If \( \psi(u) \geq \phi(u) \), then

\[
\int_u^{\phi(u)} p(t) \, dt \leq \pi p(\phi(u))^{1/2}.
\]

If \( \psi(u) < \phi(u) \), then

\[
\int_u^{\phi(u)} p(t) \, dt \leq \frac{3}{2} \pi p(\phi(u))^{1/2}.
\]

(The difference \( \phi(u) - \psi(u) \) has been investigated in [2].) Together, we obtain:

**Theorem 3.** For monotone increasing, continuous, positive \( p(t) \), the Lyapunov integral satisfies

\[
\pi^2 \left( \frac{p(u)}{p(\phi(u))} \right)^{1/2} \leq L(u) \leq \left[ 1 + \frac{1 + \epsilon}{4} \right] \pi^2 \left( \frac{p(\phi(u))}{p(u)} \right)^{1/2},
\]

\[
\epsilon = \text{sgn}[\phi(u) - \psi(u)].
\]

**References**


Polytechnic Institute of Brooklyn, Brooklyn, New York 11201