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BESSEL POTENTIALS. INCLUSION RELATIONS AMONG CLASSES OF EXCEPTIONAL SETS

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1. Let $g_\alpha = g_\alpha(x)$ be the Bessel kernel of order α, $0<\alpha<+\infty$, on R^n; g_α is the Fourier transform of $(2\pi)^{-n/2}(1+|\zeta|^2)^{-\alpha/2}$. For $1<p<\infty$, we define a capacity $B_{\alpha,p}$ (referred to as Bessel capacity): for $A\subset R^n$,

$$B(A) = B_{\alpha,p}(A) = \inf_f \int f(x)^p \, dx$$

the infimum being taken over all functions f in $L_p^+ = L_p^+(R^n)$—positive functions in the Lebesgue class—such that $g_\alpha * f(x) \geqq 1$ for all $x\in A$. The capacities $B_{\alpha,p}$ have been studied extensively in [4]. It is an easy consequence of the definition of $B_{\alpha,p}$ that: $B_{\alpha,p}(A)=0$ if and only if there is an $f\in L_p^+$ such that $g_\alpha * f(x) = +\infty$ on A.

Variants of the Bessel capacities occur for instance in [1], [3], [5].

Our purpose here is to announce results on the relations between the B's for various pairs (α, p). We say that the Bessel capacity B is *stronger* than the Bessel capacity B' (written $B'\preceq B$) if $B(A)=0$ always implies $B'(A)=0$. If in addition, there is a set A such that $B(A)>0$ but $B'(A)=0$ we say B is *strictly stronger* than B' ($B'\prec B$). These are the *relations* between B and B'. If both $B'\preceq B$ and $B\preceq B'$ hold, we say B is *equivalent* to B' ($B\sim B'$). In addition to the relations among the B's, we also give some results concerning relations between Bessel capacities, Hausdorff measures, and classical capacities (C_k below). These classical capacities can be viewed as a special case of general L_p-capacities of [4] when $p=1$ or $p=2$.

Notation. wei B = weight of $B_{\alpha,p}=\alpha p$; ord B = order of $B_{\alpha,p}=\alpha$; ind B = index of $B_{\alpha,p}=(\alpha, p)$. By $(\alpha, p)\prec(\beta, q)$ we shall mean either $\alpha p<\beta q$, or $\alpha p=\beta q$ and $\alpha<\beta$.

2. The principal result is

THEOREM 1. *If B and B' are two Bessel capacities,*
(i) *$B'\prec B$ if and only if* ind $B'\prec$ ind B *and* wei $B'\leqq n$.
(ii) *$B'\sim B$ if and only if* ind B' = ind B *or* wei B *and* wei $B'>n$.

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Let $k = k(r)$, $r = |x|$, be a nonincreasing lower semicontinuous function on $[0, +\infty)$ (referred to as a *kernel*). Define the classical capacity

$$C_k(K) = \sup_\mu \|\mu\|_1,$$

the supremum being over all positive Radon measures with finite total variation $\|\mu\|_1$ and whose supports are in the given compact set K. C_k^* denotes the usual extension of C_k to an outer capacity.

We are interested in $k(r) = g_{\alpha,q}(r) \equiv g_\alpha(r) \cdot l_q(r)$, where $l_q(r) = (\log 1/r)^{q-2}$ for $0 < r \leqq r_0 < 1$ and constantly equal to $l_q(r_0)$ for $r > r_0$. For this choice of k we denote C_k^* by $B_{(\alpha,q)}$.

THEOREM 2. (i) *If* $\alpha p = n$,

(a) $1 < p < q < \infty$ *implies* $B_{(n,q)} \preceq B_{\alpha,p}$;

(b) $1 < q < p < \infty$ *implies* $B_{\alpha,p} \preceq B_{(n,q)}$.

(ii) *If* $\alpha p < n$,

(a) $2 \leqq p < q < \infty$ *implies* $B_{(\alpha p,q)} \preceq B_{\alpha,p}$;

(b) $1 < q < p \leqq 2$ *implies* $B_{\alpha,p} \preceq B_{(\alpha p,q)}$.

The relation $B_{(\alpha p,q)} \preceq B_{\alpha,p}$ for $\alpha p < n$, $1 < p < q < 2$, is false in contrast to (i). An example showing this can easily be constructed using [6].

If $h = h(r)$ is nondecreasing with $h(0) = 0$, the Hausdorff h-measure is defined as

$$H_h(A) = \lim_{\epsilon \to 0} \inf \sum_i h(r_i)$$

the infimum being over countable covers of A by balls of radius $r_i \leqq \epsilon$, $\epsilon > 0$.

THEOREM 3. (i) *If* $\alpha p = n$,

$$B_{\alpha,p} \preceq H_h, \quad \textit{for } h(r) = (\log 1/r)^{1-p} \qquad (0 \leqq r \leqq \tfrac{1}{2}),$$

$$H_h \preceq B_{\alpha,p}, \quad \textit{for } h(r) = (\log 1/r)^{1-q}, \qquad 1 < p < q < \infty.$$

(ii) *If* $\alpha p < n$,

$$B_{\alpha,p} \preceq H_h, \quad \textit{for } h(r) = r^{n-\alpha p} \qquad (0 \leqq r \leqq r_0 < 1),$$

$$H_h \preceq B_{\alpha,p}, \quad \textit{for } h(r) = r^{n-\alpha p}(\log 1/r)^{1-q}, \qquad q > p.$$

In Theorem 1 and parts (i)(a), (ii)(a) and (b) of Theorem 2, the relation $\preceq$ can be replaced by an inequality of the type: $B'(A) \leqq QB(A)$, for all $A \subset R^n$, Q a constant independent of A. Inequalities of this type imply: $B'(A) = +\infty$ implies $B(A) = +\infty$. In part (i)(b) of Theorem 2, the Q may depend on the diameter of A. In Theorem 3,

$H_h < \infty$ is sufficient for $B=0$.

3. The two principal ingredients that go into proving Theorem 1 in the case wei $B=$ wei $B' < n$ are possibly of interest in themselves.

THEOREM 4. *Let k_1 and k_2 be kernels and μ a positive Radon measure. Define $u(x)=k_1 * (k_2 * \mu)^{1/(p-1)}(x)$; then, for all x,*

$$u(x) \leqq Q^{1/(p-1)} \sup_{y \in \operatorname{Supp} \mu} u(y), \qquad 1 < p < 2,$$

$$u(x) \leqq Q \sup_{y \in \operatorname{Supp} \mu} u(y), \qquad 2 \leqq p < \infty.$$

Here Q is a constant depending only on n, the dimension of R^n. Furthermore, in all cases $u(x) \leqq \sup_{y \in \operatorname{co}(\operatorname{Supp} \mu)} u(y)$.

THEOREM 5. *Let $f=f(x)$ be a nonnegative, extended real-valued measurable function on R^n, then*

$$\|g_{\alpha\tau} * f^t\|_r \leqq Q\|f\|_p^{t-\tau} \|g_\alpha * f\|_q^\tau,$$

with Q a constant independent of f and $0<p<\infty$, $1 \leqq q \leqq \infty$, $0<\tau<1$, $0<t-\tau<p(1-\tau)$, and $1/r=(t-\tau)/p+\tau/q$. Here $\|\cdot\|_p$ denotes the usual norm in L_p.

Details of the above theorems will appear elsewhere along with construction of the Cantor sets needed to get the relation $\prec$ in Theorem 1.

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