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KREISS' MIXED PROBLEMS WITH NONZERO INITIAL DATA

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In [3], Kreiss has shown that a large class of mixed initial boundary-value problems of hyperbolic type are well-posed in the $\mathcal{L}_2$ sense. However only zero initial data were considered. For the same class of problems we show that if square integrable initial data are prescribed then there is a unique solution which is square integrable for each positive time.

The differential operators under consideration are of the form

$$Lu = \frac{\partial u}{\partial t} - \sum_{j=1}^{n} A_j(t, x) \frac{\partial_j u}{\partial x_j} + B(t, x)u$$

where u is a complex k-vector, and A_j and B are $k \times k$ matrix-valued functions. The operator (L) is assumed strictly hyperbolic, that is $\sum A_j\xi_j$ has k distinct real eigenvalues for each $\xi \in R^n \backslash 0$. The coefficients are assumed to be smooth functions which are constant outside a compact set. In addition, we require that $\det A_1 \neq 0$ when $x_1 = 0$. The following notation is employed:

$$\Omega = \{x \in R^n \mid x_1 \geqq 0\},$$
$$\partial\Omega = \text{boundary of } \Omega = \{x \in R^n \mid x_1 = 0\},$$
$$x = (x_1, x') = (x_1, x_2, \cdots, x_n).$$

Boundary conditions are prescribed with the aid of a boundary operator $M(t, x')$ which is a smooth $l \times k$ matrix-valued function where l = number of negative eigenvalues of A_1. We suppose that M has rank l and is independent of t, x' for $|t| + |x'|$ large.

The basic problem is to show that for given

$$F \in \mathcal{L}_2([0, T] \times \Omega), \qquad g \in \mathcal{L}_2([0, T] \times \partial\Omega), \qquad f \in \mathcal{L}_2(\Omega).$$

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There is a solution u to the mixed initial boundary-value problem (I):

$$\text{(I)}\qquad \begin{aligned} Lu &= F && \text{in } [0, T] \times \Omega, \\ Mu &= g && \text{in } [0, T] \times \partial\Omega, \\ u(0, \cdot) &= f(\cdot). \end{aligned}$$

We insist that (I) is satisfied in the strong sense, that is, $u \in \mathcal{L}_2([0, T] \times \Omega)$ and there exist functions $u^n \in C_0^\infty(R \times \Omega)$ with (all norms are $\mathcal{L}_2$ norms)

$$\|u^n - u\|_{[0, T] \times \Omega} \to 0, \qquad \|Lu^n - F\|_{[0, T] \times \Omega} \to 0,$$

$$\|Mu^n|_{x_1=0} - g\|_{[0, T] \times \partial\Omega} \to 0, \qquad \|u^n(0, \cdot) - f(\cdot)\|_{\Omega} \to 0.$$

In the case of constant coefficient problems with $B=0$, Hersh [2] has given necessary and sufficient conditions for (I) to have a unique smooth solution for all smooth data which vanish in a neighborhood of $\{(t, x) | t=0=x_1\}$. This type of solubility is weaker than the $\mathcal{L}_2$ well-posedness described above. We impose

Kreiss' condition. For each point $(t, 0, x')$, the constant coefficient problem that arises by freezing A_j and M at this point and setting $B=0$ is well-posed in the sense of Hersh for all $l \times k$ matrices in a neighborhood of M.

THEOREM. *If M satisfies Kreiss' condition then* (I) *has a unique strong solution u for arbitrary $F \in \mathcal{L}_2([0, T] \times \Omega)$, $g \in \mathcal{L}_2([0, T] \times \partial\Omega)$, $f \in \mathcal{L}_2(\Omega)$. After modification on a set of measure zero in $[0, T] \times \Omega$ we have $u(t, \cdot) \in \mathcal{L}_2(\Omega)$ for $0 \leqq t \leqq T$ and*

$$\text{(1)}\qquad \|u(t, \cdot)\|_{\Omega} \leqq c(\|F\|_{[0, t] \times \Omega} + \|g\|_{[0, t] \times \partial\Omega} + \|f\|_{\Omega})$$

with c independent of F, g, f, t.

There are two extensions of this result which present no real difficulty. The first is when Ω is a region with smooth noncharacteristic boundary. Standard techniques employing a partition of unity and a change of local coordinates reduce this to the problems treated above. The second is regularity of solutions. If F, g, f have square integrable derivatives up to order m then so does u provided F, g, f satisfy appropriate compatibility conditions in a neighborhood of $t=0$, $x_1=0$.

INDICATION OF PROOF. The crucial ingredient is that when $f=0$ Kreiss has shown that (I) is solvable and given estimates for the solution *and its boundary values*, that is,

$$\|u\|_{[0,\,t]\times\Omega} + \|u|_{x_1=0}\|_{[0,\,t]\times\partial\Omega} \leqq c(\|F\|_{[0,\,t]\times\Omega} + \|g\|_{[0,\,t]\times\Omega}). \tag{2}$$

This allows us to integrate by parts (using the method of Garding and Leray [1]) with estimates for all contributions. In this way an estimate of the form

$$\|u(t,\,\cdot)\|_{\Omega} \leqq c(\|F\|_{[0,\,t]\times\Omega} + \|g\|_{[0,\,t]\times\Omega}) \tag{3}$$

can be derived in a high Sobolev norm. It is then shown that when $f \neq 0$ an inequality of the form

$$\begin{aligned} &\|u\|_{[0,\,T]\times\Omega} + \|u|_{x_1=0}\|_{[0,\,T]\times\partial\Omega} \\ &\qquad \leqq c(\|F\|_{[0,\,T]\times\Omega} + \|g\|_{[0,\,T]\times\partial\Omega} + \|f\|_{\Omega}) \end{aligned} \tag{4}$$

in an appropriate negative norm is "dual" to (2) and (3). Using the fact that (4) also holds for the derivatives of u the negative norms can be "raised" to norms in the spaces H_s with $s \geqq 0$. Another integration by parts yields (1) in a high Sobolev norm. Then a dual inequality and "raising" argument finish the proof.

When the A_j are hermitian simpler proofs are available (see [4]). Heinz Kreiss' active participation in this research is gratefully acknowledged.

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