EXISTENCE OF POLYNOMIAL IDENTITIES IN $A \otimes_F B$

BY AMITAI REGEV

Communicated by M. H. Protter, April 28, 1971

ABSTRACT. The following theorem is proved: If $A$, $B$ are PI-algebras over a field $F$, then $A \otimes_F B$ is also a PI-algebra.

Let $F$ be a field, $A$ and $B$ two PI-algebras (i.e., algebras satisfying a polynomial identity) over $F$. The problem whether also $A \otimes_F B$ satisfies a polynomial identity has been open for some time [1, p. 228]. We have proved that if $A$ and $B$ are PI-algebras, then $A \otimes_F B$ is indeed a PI-algebra. A very brief outline of the proof is given here, and the details of the proof will appear elsewhere.

Let $\{x\}$ be an infinite set of noncommutative indeterminates over $F$, and let $F[\{x\}]$ be the free ring in $\{x\}$ over $F$. Let $\{x_1, x_2, \cdots\}$ be a fixed countable sequence of indeterminates from $\{x\}$. Let $S_n$ denote the group of all permutations of $\{1, \cdots, n\}$ and let

$$V_n = \text{span}\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

be the $n!$ dimensional vector space, spanned by the $n!$ monomials $x_{\sigma(1)} \cdots x_{\sigma(n)}$ $(\sigma \in S_n)$ in $x_1, \cdots, x_n$.

An ideal $Q \subseteq F[\{x\}]$ is a $T$-ideal if $f(x_1, \cdots, x_n) \in Q$ and $g_1, \cdots, g_n \in F[\{x\}]$ implies that $f(g_1, \cdots, g_n) \in Q$. It is well known [1, p. 234] that the set of all identities of a PI-algebra is a $T$-ideal. Let $Q$ be the $T$-ideal of identities of a PI-algebra $A$. For each integer $0 < n$, define $d_n = \dim(V_n/(Q \cap V_n))$. We call $\{d_n\}$ “the sequence of codimensions” of $Q$ (or $A$). Codimensions play an important role in the proof that $A \otimes_F B$ is a PI-algebra.

It follows from the definition of $d_n$ that there exist $d_n$ monomials $M_1(x_1, \cdots, x_n), \cdots, M_{d_n}(x_1, \cdots, x_n)$ which span $V_n$ modulo $Q$, i.e., for each $\sigma \in S_n$ there exist coefficients $\phi_\sigma(\sigma) \in F$, $1 \leq i \leq d_n$, such that

$$M_\sigma(x) = x_{\sigma(1)} \cdots x_{\sigma(n)} = \sum_{i=1}^{d_n} \phi_\sigma(\sigma) M_i(x) \pmod{Q}.$$
Since \( Q \) is the ideal of identities of \( A \), it follows that for any substitution \( a_1, \ldots, a_n \in A \) we have

\[
a_{a_1} \cdots a_{a_n} = \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(a_1, \ldots, a_n).
\]

We now prove

**Theorem 1.** Let \( A \) and \( B \) be two PI-algebras with \( \{d_n\}, \{h_n\} \) the corresponding sequences of codimensions. If there exists an integer \( 0 < n \) such that \( d_n h_n < n! \), then \( A \otimes B \) satisfies a nontrivial identity of degree \( n \).

**Proof.** Let \( M_1(x), \ldots, M_{d_n}(x), \phi_i(\sigma) \in F, \ 1 \leq i \leq d_n, \ \sigma \in S_n \), be monomials and coefficients such that for all \( a_1, \ldots, a_n \in A \) and \( \sigma \in S_n \), \( a_{\sigma} = \sum_{i=1}^{d_n} \phi_i(\sigma) M_i(a) \). Let, similarly, \( N_j(x), \psi_j(\sigma), \ 1 \leq j < h_n, \ \sigma \in S_n \), be monomials and coefficients such that for all \( \sigma \in S_n \) and \( b_1, \ldots, b_n \in B \),

\[
b_{\sigma_1} \cdots b_{\sigma_n} = \sum_{j=1}^{h_n} \psi_j(\sigma) N_j(b).
\]

Write

\[
f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma_1} \cdots x_{\sigma_n}
\]

with the \( \alpha_\sigma \) undetermined coefficients. Now

\[
f(a_{1 \otimes b_1}, \ldots, a_{n \otimes b_n}) = \sum_{\sigma \in S_n} \alpha_\sigma (a_{\sigma_1} \cdots a_{\sigma_n}) \otimes (b_{\sigma_1} \cdots b_{\sigma_n})
\]

\[
= \sum_{i=1}^{d_n} \sum_{j=1}^{h_n} \phi_i(\sigma) \psi_j(\sigma) \alpha_{\sigma} M_i(a) \otimes N_j(b).
\]

Since \( d_n h_n < n! \), there exists a nontrivial solution \( \{\alpha_\sigma\}_{\sigma \in S_n} \) for the \( h_n d_n \) homogeneous linear equations \( \sum_{\sigma \in S_n} \phi_i(\sigma) \psi_j(\sigma) \alpha_\sigma = 0 \) in \( n! \) indeterminates. Clearly the \( \alpha_\sigma \) yield (for (*) a nontrivial identity \( f(x_1, \ldots, x_n) \) for \( A \otimes B \).

The second and the difficult step in the proof that \( A \otimes B \) is a PI-algebra is:

**Theorem 2.** Let \( \{d_n\} \) be the sequence of codimensions of an arbitrary PI-algebra \( A \). Then there exists a positive real number \( k \) such that for all \( n \in \mathbb{N}, d_n \leq k^n \). (We actually prove if \( A \) satisfies an identity of degree \( d \), then \( k \leq 3 \cdot 4^{d-1} \).)

The proof of Theorem 2 is complicated and will be given elsewhere.
It is a combinatorial proof, and has nothing to do with the structure of the algebra.

Now, let $A, B$ be two PI-algebras with $\{d_v\}, \{h_v\}$ their corresponding sequences of codimensions. Let $k, l$ be such that for all $n$, $d_n \leq k^n$, $h_n \leq l^n$. Let $n$ be such that $(k \cdot l)^n < n!$. Then, by Theorem 1, $A \otimes B$ satisfies an identity of degree $n$.

REFERENCES


Tel-Aviv University, Tel-Aviv, Israel