A STRUCTURE THEOREM FOR COMPLETE NONCOMPACT
HYPERSURFACES OF NONNEGATIVE CURVATURE

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Communicated by I. Singer, May 20, 1971

The convexity theorem of Sacksteder-van Heijenoort [4] states
that if $M$ is a $C^\infty$ $n$-dimensional ($n > 1$) complete orientable Riemannian
manifold of nonnegative sectional curvature with positive curvature
at one point, then every isometric immersion $x: M \to \mathbb{R}^{n+1}$ is an
imbedding and $x(M)$ bounds an open convex subset of $\mathbb{R}^{n+1}$; furthermore $M$ is diffeomorphic to either $\mathbb{R}^n$ or $S^n$ (unit $n$-sphere). The
purpose of this note is to announce a structure theorem that complements the above result of Sacksteder and van Heijenoort. Full details
will appear in a forthcoming monograph on convexity and rigidity of hypersurfaces.

**THEOREM.** Let $M$ be a $C^\infty$ hypersurface in $\mathbb{R}^{n+1}$ ($n > 1$) which is complete, noncompact, orientable with nonnegative sectional curvature, which
in addition all positive at one point, then:

1. The spherical image of $M$ in the unit sphere $S^n$ has a geodesically convex closure, which lies in a closed hemisphere.
2. The total curvature of $M$ (cf. Chern-Lashof [2]) does not exceed one.
3. $M$ is a pseudograph (see below for definition) over one of its tangent planes.
4. $M$ has infinite volume.

**COROLLARY.** Suppose the sectional curvature of $M$ is in fact everywhere positive, then:

5. The spherical map is a diffeomorphism onto a geodesically convex open subset of $S^n$. Consequently the spherical image lies in an open hemisphere.
6. Coordinates in $\mathbb{R}^{n+1}$ can be so chosen that $M$ is tangent to the hyperplane $x_{n+1} = 0$ at the origin, and there is a nonnegative strictly convex function (i.e. its Hessian is everywhere positive definite) $f(x_1, \cdots, x_n)$ defined in a convex domain of $\{x_{n+1} = 0\}$ such that $M$ is exactly the graph of $f$.

**REMARKS.** (A) A $C^\infty$ convex hypersurface $M$ (i.e. $M$ is the full boundary of an open convex set) in $\mathbb{R}^{n+1}$ is said to form a pseudograph

AMS 1969 subject classifications. Primary 5374; Secondary 5375.

1 Sloan Fellow. Also partially supported by the National Science Foundation.
over the tangent plane \( H \) if and only if:

(a) \( M \) lies above \( H \), i.e. designating a closed half-space of \( H \) as being above \( H \), we have that \( M \) lies in this half-space.

(b) Let \( \pi : \mathbb{R}^{n+1} \rightarrow H \) be the orthogonal projection and let \( A = \pi(M) \). Then over the interior \( A^\circ \) (of \( A \) as a subset of \( H \)), \( M \) is the graph of a \( C^\infty \) function.

(c) For every \( a \in A - A^\circ \), \( M \cap \pi^{-1}(a) \) is a closed semi-infinite straight line segment.

(d) Every hyperplane strictly above \( H \) intersects \( M \) at a diffeomorph of the unit \((n-1)\)-sphere \( S^{n-1} \).

(B) When \( n = 2 \) and the curvature of \( M \) is everywhere positive, this theorem (as well as the theorem of Sacksteder-van Heijenoort) was first proved by Stoker [5].

(C) Assertions (2)-(6) above all follow from assertion (1). We actually prove a more general result than (1):

**Proposition.** Let \( C \) be an open convex subset of \( \mathbb{R}^{n+1} \) \((n \geq 1)\) with connected boundary \( M \) and let \( \gamma : M \rightarrow S^n \) be the spherical map (in the sense of Alexandrov, see Busemann [1]). Then the closure of \( \gamma(M) \) is geodesically convex.

The proof of this Proposition is achieved quite simply by employing the concept of the barrier cone of a convex set. See Rockafellar [3].

(D) Neither (1) nor the Proposition is true if the word “closure” is deleted. (Cf. Busemann [1, p. 25, (4.4)].)

(E) The Proposition has applications in the theory of convex surfaces, e.g. Alexandrov’s theory of spherical measures on an open convex surface (Busemann [1, p. 31]) or the rigidity and nonrigidity theorems of Pogorelov and Olovyanishnikov on open convex surfaces (Busemann [1, pp. 167–168]).

**Bibliography**


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