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BSJ DOES NOT MAP CORRECTLY INTO *BSF* MOD 2

BY ROBERT CLOUGH AND JAMES STASHEFF

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It has been widely conjectured that there is a homotopy commutative diagram

$$\text{(A)}\qquad \begin{array}{ccc} BSO & \xrightarrow{\;J\;} & BSF \\ & j\searrow \quad \nearrow & \\ & BSJ & \end{array}$$

where J is the stable Whitehead J-homomorphism and BSJ is the space constructed in [1]. Indeed, in [5], Quillen proves the Adams conjecture, which implies that SJ maps into SF mod 2 in a way consistent with diagram (A). In [6], Sullivan proves that SF even splits mod 2 into $SJ\times \text{Coker}(J)$, although this splitting does not necessarily deloop to a splitting of BSF. In [3], Madsen proves that diagram (A), if it exists, does not deloop twice.

Our purpose is to sketch a proof that diagram (A) does not exist. Details will follow in [2].

THEOREM. *It is impossible to define Stiefel-Whitney classes w_n for $n\geqq 2$ in $H^n(BSJ;Z_2)$ in such a way that all of the following conditions hold*:

(1) $j^*w_n = w_n \in H^n(BSO;Z_2)$;

(2) *the w_n satisfy the Wu formulas.*

COROLLARY. *Diagram* (A) *does not exist* mod 2.

SKETCH OF PROOF OF THEOREM. We assume for a contradiction that Stiefel-Whitney classes can be chosen satisfying conditions (1) and (2).

By [1], we have

$$H^*(BSJ;Z_2) = P[w_2, w_3, \cdots] \otimes E[e_3, e_4, \cdots]$$

and BSJ is the base of the 2-primary fibration

$$BSO \xrightarrow{\psi^3 - 1} BSO \xrightarrow{j} BSJ.$$

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The classes $e_3, e_4, \cdots$ are the images of classes in $H^*(BBSO; Z_2)$.

By the assumed operation of the Steenrod algebra on $H^*(BSJ; Z_2)$ and by the second order Bockstein relation

$$d_2 x^2 = xSq^1x + Sq^{2n}Sq^1x$$

whenever degree $(x) = 2n$, we see that the E_3 level of the Bockstein spectral sequence is

$$P[x_n \mid n \geqq 1] \otimes E[y_{n+1} \mid n \geqq 1] \otimes E[a_m \mid m = 2^k \geqq 8]$$

where

$$\begin{aligned} x_n &= w_{2n}^2; & \\ y_{n+1} &= e_{4n+1} & &\text{if } n \text{ is a power of 2,} \\ &= e_{2n}e_{2n+1} & &\text{otherwise;} \\ a_m &= e_3e_4e_8 & &\text{if } k = 3, \\ &= e_m & &\text{if } k \geqq 4. \end{aligned}$$

We now construct the diagram of Peterson and Toda [4]

$$\begin{array}{ccc} C_\lambda & \xrightarrow{f} & BSO \\ h\downarrow & & \downarrow j \\ SRP & \xrightarrow[g]{} & BSJ \end{array}$$

in which $\lambda: RP \to CP$ is the nontrivial map. The integral cohomology of C_λ is as follows: $H^{2n}(C_\lambda; Z) = Z$ generated by z_{2n}; $H^{2n+1}(C_\lambda; Z) = 0$. The z_{2n} map to $2\alpha^n$ in $H^{2n}(CP; Z)$.

The maps f and g can be picked so that f^* and g^* annihilate w_{12} and e_{12} mod 2 but so that $f^*P_3 = 4z_{12}$ mod Z, where P_3 is the Pontryagin class. The class w_6^2 of BSJ can then be lifted mod 8 to a class k' such that

$$j^*k' = r_8(P_3 + 2p(P_1, P_2)) + r_{2,8}(t)$$

where p is a polynomial, r_8 is reduction mod 8, $r_{2,8}$ is induced by the nontrivial map $Z_2 \to Z_8$, and $t \in H^{12}(BSO; Z_2)$. We can modify k' to get a class k such that

$$j^*k = r_8(P_3 + 2p(P_1, P_2)).$$

Then g^*k^* is nontrivial mod 8, so that kg is the nontrivial map

$$SRP \to K(Z_8, 12).$$

Thus $(kg)^*b_{13} \neq 0$ mod 2, where b_{13} is the generator of $H^{13}(K(Z_8, 12); Z_2)$.

But this is impossible since g^* must annihilate all decomposables and since it was chosen so as to annihilate $w_{13}=Sq^1w_{12}$ and $e_{13}=Sq^1e_{12}$, the "only" indecomposable elements of $H^{12}(BSJ;\, Z_2)$.

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UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556

TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19122