TAMING IRREGULAR SETS OF HOMEOMORPHISMS

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1. Introduction. Let $\mathcal{U}$ be an $n$-dimensional open connected manifold, $\mathcal{U}^\infty = \mathcal{U} \cup \{\infty\}$ the one-point compactification of $\mathcal{U}$, and $d$ a metric on $\mathcal{U}^\infty$. Suppose that $h$ is a homeomorphism of $\mathcal{U}$ onto itself and let $h_\infty$ be the extension of $h$ to $\mathcal{U}^\infty$. If $p \in \mathcal{U}^\infty$, we say that $h$ is regular at $p$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(p, q) < \delta$ implies that $d(h_\infty^n(p), h_\infty^n(q)) < \varepsilon$ for all $n$. If $h$ is not regular at $p$, we say that $p$ is an irregular point of $h$.

Homeomorphisms with finitely or countably many irregular points have been studied extensively \[4\]-\[10\], \[12\]. In this paper, we consider homeomorphisms $h$ which satisfy

1. the set of irregular points of $h$ is $P \cup \{\infty\}$, where $P$ is a $k$-dimensional continuum with $k \leq n - 2$,

and seek conditions on $h$ which imply that $P$ is nicely embedded. Details of proofs will appear elsewhere.

2. Nice homeomorphisms. Suppose that $\mathcal{U}$ and $h$ are as above. We say that $h$ is a nice homeomorphism if $h$ satisfies (1),

2. for each $x \in \mathcal{U} - P$, $\lim_{n \to \infty} h^n(x) \in P$ and $\lim_{n \to -\infty} h^n(x) = \infty$, and

3. the mapping $f_h: \mathcal{U} \to P$ given by $f_h(x) = \lim_{n \to -\infty} h^n(x)$ exists and is continuous.

Remarks. If $h$ satisfies (1), the work of T. Homma and S. Kinoshita \[5\] can be used to show that either $h$ or $h^{-1}$ satisfies (2), so that the strength of our assumptions is in (3). For example, let $h: S^1 \times R^2 \to S^1 \times R^2$ be defined by $h(x, t) = (k(x), \frac{1}{2}t)$ where $k: S^1 \to S^1$ is rotation through an irrational multiple of $\pi$ radians. Then $h$ satisfies (1) and (2) with $P = S^1 \times \{0\}$, but $h$ does not satisfy (3).

The canonical example of a nice homeomorphism is the case where $\mathcal{U}$ is an open mapping cylinder over $P$ and $h$ is a homeomorphism which "pushes in" along the product structure.

Proposition 1. If $h$ is a nice homeomorphism, then

(i) $P$ is an absolute neighborhood retract;

(ii) $f_h$ is onto;

(iii) the fixed point set of $h$ is $P$;

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(iv) the inclusion \( P \subseteq \mathcal{U} \) is a homotopy equivalence;
(v) the natural projection \( p \) of \( \mathcal{U} - P \) onto the orbit space \( \hat{\mathcal{U}} \) of \( h|\mathcal{U} - P \) is a covering map;
(vi) \( \mathcal{U} \) is a closed \( n \)-manifold; and
(vii) \( f_h \) induces a map \( \hat{f}_h : \hat{\mathcal{U}} \to P \) such that \( \hat{f}_hp = f_h \).

(i)-(iv) follow from point set arguments and the fact that \( hf_h = f_h \).
(v)-(vii) follow from elementary facts about covering spaces and [11].

3. AFG sets and maps. If \( X \) is a continuum in the ENR \( M \), we say that \( X \) has property AFG if there is a neighborhood \( W \) of \( X \) in \( M \) such that for each neighborhood \( U \) of \( X \) in \( W \) there is a neighborhood \( V \) of \( X \), \( V \subset U \) such that each map of \( S^1 \) into \( V \) which is null homologous in \( U \) is null homotopic in \( U \).

It can be shown, in the spirit of [13], that the AFG property depends only on the homotopy type of \( X \).

If \( f \) is a proper map between manifolds, we say that \( f \) is an AFG map provided that \( f^{-1}(x) \) has property AFG for each \( x \) in the image of \( f \).

4. Taming irregular sets in high dimensions. If \( P \) is a polyhedron in \( \mathcal{U} \), we say that \( P \) is locally flat if \( P \) has a triangulation in which each simplex is locally flat.

**Theorem 2.** If \( h \) is a nice homeomorphism with \( P \) a polyhedron, \( n \geq 6 \), and \( k + 3 \leq n \), then \( P \) is locally flat if and only if \( \hat{f}_h \) is an AFG map.

Theorem 2 is proven by using the homotopy properties of \( \hat{f}_h \) to show that \( P \) is locally nice and by applying Bryant and Seebeck [3]. An important step in the proof is the application of L. Siebenmann’s obstruction theory [15] to prove

**Theorem 3.** If \( \hat{f}_h \) is AFG and \( B \) is the open star of some point in \( P \) in some triangulation of \( P \), then \( \hat{f}_h^{-1}(B) \) is homeomorphic to the interior of a compact manifold provided \( n \geq 6 \).

5. The three-dimensional case. If \( h \) is a nice homeomorphism, we say that \( h \) has a cross-section if there is a closed, locally flat \( (n - 1) \)-manifold \( T \subset \mathcal{U} - P \) such that \( f_h^{-1}(x) \cap T \) is a continuum for each \( x \in P \), \( T \) separates \( \mathcal{U} \) into two components with \( P \) in the bounded component, and \( h(T) \cap T = \emptyset \).

**Theorem 4.** Let \( h \) be a nice homeomorphism with cross-section, \( n = 3 \), and \( k = 1 \). Then \( P \) is locally tame at each point and \( \mathcal{U} \) is homeomorphic to the interior of a cube with \( q \) handles, where \( q = \text{rank } H_1(P) \).

The proof of Theorem 4 is a lengthy argument using standard tools in three-dimensional topology. An important step in the proof involves an appeal to a taming theorem of D. R. McMillan [14].
If \( p \in U \), we say that \( h \) is positively regular at \( p \) if for each \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( d(p, q) < \delta \) implies \( d(h^n(p), h^n(q)) < \epsilon \) for all \( n > 0 \).

**Proposition 5.** If \( h \) satisfies (1) and (2), \( k = 1, P \neq S^1, h|P = \text{identity}, \) and \( h \) is positively regular on \( U \), then \( h \) is a nice homeomorphism.

Theorem 4, then, has an obvious restatement in terms of positive regularity. Examples can be given to show that Theorem 4 cannot be extended to higher dimensions. In fact, the construction of M. Brown [2] using the Andrews-Curtis Theorem [1] can be used to construct, for each \( n \geq 4 \) and \( 1 \leq k \leq n - 3 \), a homeomorphism \( h \) which satisfies (1) and (2) with \( U = \mathbb{R}^n \) and \( P \) a wildly embedded \( k \)-cell, such that \( h \) has a cross-section and is positively regular on \( \mathbb{R}^n \).

**References**


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