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QUASI-ANALYTICITY AND SEMIGROUPS OF BOUNDED LINEAR TRANSFORMATIONS

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Suppose H is a real Banach space and T is a strongly continuous (on $[0, \infty)$) semigroup of bounded linear transformations from H to H. Steps leading to the following will be indicated:

THEOREM. *If*

$$\liminf_{x \to 0} |T(x) - I| < 2 \tag{1}$$

then the set of all functionals of trajectories of T form a quasi-analytic collection.

COROLLARY. *If* (1) *is satisfied, then $T(x)$ is invertible for all $x > 0$ (although $(T(x))^{-1}$ may be unbounded).*

A functional of a trajectory of T is a function h with domain $(0, \infty)$ for which there is f in H^* and p in H so that $h(x) = f(T(x)p)$ for all $x > 0$. A collection G of real-valued functions with a common connected domain J is quasi-analytic provided no two members of G agree on an open subset of J.

In [7] it is shown that if

$$\limsup_{x \to 0} |T(x) - I| < 2, \tag{2}$$

then every functional of a trajectory of T is real-analytic (and $AT(x)$ is bounded for all $x > 0$ where A is the generator of T). An example in [7] can be used to show that (1) does not imply (2).

Recent closely related results [**1**], [**3**], [**8**], [**9**], [**2**] as well as [**7**] connect the following: (a) the degree of approximation of the identity by the semigroup, (b) properties of the generator and (c) regularity properties of trajectories. For T a Markov semigroup it may be seen from [**4**] that (1) follows from a condition on transition probabilities ($\Gamma > 0$).

Lemmas 1 and 2 which follow are improvements of Lemma 7 of [**6**] and Theorem 1 of [**5**] respectively.

Suppose f is a real-valued continuous function with domain $[0, 1]$ so that $f(x) = 0$ if $0 \leq x \leq \frac{1}{2}$ and, if $y > \frac{1}{2}$, then there is a number x in $(\frac{1}{2}, y)$

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so that $f(x) \neq 0$. Suppose furthermore that $\Delta = \{\delta_q\}_{q=1}^{\infty}$ is a sequence of positive numbers converging to 0. If Q is an open interval containing 1 and q is a positive integer for which there exists a positive integer n so that $n\delta_q \in Q \cap [0, 1]$, then denote by $z(q, Q)$ the set of all such n, denote by $n(q, Q)$ the largest element of $z(q, Q)$ and denote

$$\sup\left\{\left|\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}f(r\delta_q)\right| : n \in z(q, Q)\right\}$$

by $F(q, Q)$.

LEMMA 1. *If Q is an open interval containing* 1, *then*

$$\lim_{q\to\infty} F(q, Q)^{1/n(q,Q)} = 2.$$

LEMMA 2. *Suppose J is a connected nondegenerate set of numbers. Denote by $G(\Delta, J)$ the collection of all continuous real-valued functions f with domain J and having the following property*:

If x is in J, there is an open interval S containing x and positive numbers M, ε so that if $u, v \in S \cap J$ and $|u - v| = n\delta_q$ for positive integers n and q, then

$$\left|\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}f(u + r\delta_q)\right| \leqq M(2-\varepsilon)^n.$$

Then $G(\Delta, J)$ is a quasi-analytic collection.

INDICATION OF PROOF OF THEOREM. Denote by $\{\delta_q\}_{q=1}^{\infty}$ a sequence of positive numbers converging to 0 and by ε a positive number so that $|T(\delta_q) - I| \leqq 2 - \varepsilon$, $q = 1, 2, \ldots$. Denote by w a positive number. Denote by L a number so that $|T(x)| \leqq L$ for all x in $[0, w]$.

Suppose p is in H, f is in H^* and $h(x) = f(T(x)p)$ for all $x > 0$. If each of n, q is a positive integer, $w \geqq u, v > 0$, and $|u - v| = n\delta_q$, then

$$\begin{aligned}\left|\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}h(u + r\delta_q)\right| &= \left|f\left[\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}T(u + r\delta_q)p\right]\right| \\ &= \left|f\left[\left(\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}(T(\delta_q))^r\right)T(u)p\right]\right| \\ &= |f[(T(\delta_q) - I)^n T(u)p]| \\ &\leqq |f|L\|p\|\,|T(\delta_q) - I|^n \leqq |f|L\|p\|(2-\varepsilon)^n.\end{aligned}$$

This is sufficient to place h in $G(\Delta, (0, \infty))$.

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