BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

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NORMAL SOLVABILITY AND ϕ-ACCRETIVE MAPPINGS OF BANACH SPACES[1]

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Communicated August 2, 1971

Let X and Y be real Banach spaces. A mapping f of X into Y is said to be *normally solvable* if $f(X)$ is closed in Y. The theory of normal solvability uses this property together with infinitesimal assumptions upon the structure of $f(X)$ to obtain conclusions upon the global structure of $f(X)$, and in particular the conclusion that $f(X)$ is all of Y or that a given element y_0 of Y lies in $f(X)$.

It is our purpose in the present note to present some new and sharper results in this theory, and to apply these results to the proof of existence theorems for equations of the form $f(x) = y$ for mappings f which lie in a general class of ϕ-accretive mappings, generalizing the concept of a monotone mapping from X to X^* and of an accretive mapping from X to X.

DEFINITION 1. *Let X and Y be real Banach spaces, Y^* the conjugate space of Y. Let ϕ be a mapping of X into Y^* such that $\phi(X)$ is dense in Y^* with*

$$\|\phi(x)\|_{Y^*} = \|x\|_X, \qquad \phi(\xi x) = \xi\phi(x),$$

for all x in X, $\xi \geqq 0$. Then:

(a) *A mapping f of X into Y is said to be ϕ-accretive if, for all x and u in X,*

$$(f(x) - f(u), \phi(x - u)) \geqq 0.$$

(b) *The map f is said to be strongly ϕ-accretive if there exists $c > 0$ such that, for all x and u in X,*

$$(f(x) - f(u), \phi(x - u)) \geqq c\|x - u\|^2.$$

AMS 1970 *subject classifications*. Primary 47H15; Secondary 47H05.

(c) *The map f is said to be strictly ϕ-accretive if, for all x and u in X with $x \neq u$,*

$$(f(x) - f(u), \phi(x - u)) > 0.$$

(d) *The mapping f is said to be firmly ϕ-accretive if there exists a continuous strictly increasing function c from R^+ to R^+ with $c(0) = 0$ such that*

$$(f(x) - f(u), \phi(x - u)) \geqq c(\|x - u\|) \qquad (x, u \in X).$$

Similar definitions may be formulated for maps f of X into 2^Y, and in particular for single-valued mappings f defined only on a subset $D(f)$ of X.

We shall derive existence theorems for ϕ-accretive mappings from general results of the normal solvability theory. Two such results which are stated in [**6**] and proved in [**7**] and [**8**] are the following:

THEOREM 1. *Let X and Y be Banach spaces, f a mapping of X into Y, y_0 a point of Y. Suppose that $f(X)$ is closed in Y and that there exist constants $r > 0$, $p < 1$ such that the following conditions hold:*

(a) $B_r(y_0) \cap f(X) \neq \varnothing$.

(b) *For each y in $f(X) \cap B_r(y_0)$, there exists a sequence $\{y_j\}$ in $f(X)$ with $y_j \neq y$ for each j such that $y_j \to y$, and a sequence $\{\xi_j\}$ of nonnegative real numbers such that, for each j,*

$$\|\xi_j(y_j - y) - (y_0 - y)\| \leqq p\|y_0 - y\|.$$

Then, y_0 lies in $f(X)$.

THEOREM 2. *Let X and Y be Banach spaces with Y^* uniformly convex, f a mapping of X into Y, y_0 a point of Y. Let J be the (normalized) duality mapping of Y into Y^* which is uniquely defined by $\|J(y)\| = \|y\|$, $(J(y), y) = \|y\|^2$ for each y in Y. Suppose that $f(X)$ is closed in Y and that there exist $r > 0$, $\delta > 0$ such that the following two conditions hold:*

(a) $B_r(y_0) \cap f(X) \neq \varnothing$.

(b) *For each y in $B_r(y_0) \cap f(X)$, there exists a sequence $\{y_j\}$ in $f(X)$ such that $y_j \to y$, $y_j \neq y$ for each j, and such that*

$$(J(y_0 - y), y_j - y) \geqq \delta\|y_j - y\| \cdot \|y_0 - y\|.$$

Then, y_0 lies in $f(X)$.

We now prove a sharper version of Theorem 2 under a still stronger structural hypothesis on the image space Y, and then apply these results to the existence theory of ϕ-accretive mappings.

THEOREM 3. *Let X and Y be Banach spaces with Y uniformly convex and Y having its norm twice-differentiable on the unit sphere with bounded* *second derivative, so that the duality mapping J of Y into Y^* satisfies the Lipschitz condition*

$$\|J(y_1) - J(y_2)\| \leqq M\|y_1 - y_2\| \qquad (y_1, y_2 \in Y).$$

Suppose that $f(X)$ is closed in Y, and that for each y_0 in Y, the following two conditions hold:

(a) *There exists $r > 0$ such that $f(X) \cap B_r(y) \neq \emptyset$.*

(b) *If y_0 does not lie in $f(X)$, then for each y in $B_r(y_0) \cap f(X)$, there exists v in $f(X)$ such that*

$$(J(y_0 - y), v - y) > M\|v - y\|^2.$$

Then, $f(X)$ is all of the space Y.

PROOF OF THEOREM 3. Since $f(X)$ is closed in Y, it suffices to prove $f(X)$ dense in Y. Let

$$Y_0 = \{y_0 | y_0 \in Y, \text{there exists } y \text{ in } f(X) \text{ such that } \|y_0 - y\| = \operatorname{dist}(y_0, f(X))\}.$$

By the theorem of Edelstein [**9**], Y_0 is dense in Y. Hence, it suffices to prove that $Y_0 \subset f(X)$.

Let y_0 be a point of $Y_0 - f(X)$, and let y be a point of $f(X)$ such that $\|y_0 - y\| \leqq \|y_0 - v\|$ for all v in $f(X)$. Since the duality mapping J is the subgradient of the convex function $g(y) = \frac{1}{2}\|y\|^2$ on Y, we see that

$$g(v - y_0) \geqq g(y - y_0) \geqq g(v - y_0) + (J(y_0 - v), v - y).$$

Hence

$$(J(y_0 - v), v - y) \leqq 0.$$

As a consequence,

$$(J(y_0 - y), v - y) \leqq \|J(y_0 - y) - J(y_0 - v)\| \cdot \|v - y\| \leqq M\|v - y\|^2.$$

The last inequality is true for all v in $f(X)$. By hypothesis, the reverse inequality is true for some v in $f(X)$ if y_0 does not lie in $f(X)$. Hence $Y_0 \subset f(X)$, and hence $f(X) = Y$. Q.E.D.

THEOREM 4. *Let X and Y be Banach spaces, f a strongly ϕ-accretive mapping of X into Y. Suppose that one of the two following additional hypotheses holds:*

(I) *Y^* is uniformly convex and f is locally Lipschitzian.*

(II) *Y and Y^* are uniformly convex, J satisfies a Lipschitz condition, $\phi(X) = Y^*$, and f satisfies the following* lip($\frac{1}{2}$)*-condition:*

$$\|x - u\|^{-1/2}\|f(x) - f(u)\| \to 0, \quad \text{as } \|x - u\| \to 0,$$

for each u in X.

Then, $f(X) = Y$.

PROOF OF THEOREM 4. Since f is strongly ϕ-accretive, we have

$$c\|x-u\|^2 \leqq (f(x)-f(u), \phi(x-u)) \leqq \|f(x)-f(u)\| \cdot \|\phi(x-u)\| \leqq \|f(x)-f(u)\| \cdot \|x-u\|.$$

It follows that $\|f(x)-f(u)\| \geqq c\|x-u\|$. Since f is continuous, it is an immediate consequence that $f(X)$ is closed in Y. To prove that $f(X) = Y$, it therefore suffices by the connectedness of Y to prove that $f(X)$ is open in Y.

PROOF FOR CASE (I). Let y_1 be a point of $f(X)$. We consider a point y_0 of $B_r(y_1)$ such that f satisfies a Lipschitz condition with constant M_0 on $B_{2r}(y_1)$. We wish to prove that y_0 lies in $f(X)$, by applying the result of Theorem 2. Let y be a point of $B_r(y_0) \cap f(X)$, $y = f(u)$. We apply the strong accretiveness condition with respect to the points x_t and u, where $x_t = u + tv$, $t > 0$, and obtain

$$t(f(x_t)-f(u), \phi(v)) \geqq ct^2\|v\|^2.$$

We choose an element v of X with $\|v\| = \|y_0 - y\|$ such that $\|\phi(v) - J(y_0 - y)\| < \varepsilon$ for a suitable small $\varepsilon > 0$, to be chosen in a moment. Then

$$(f(x_t)-f(u), J(y_0-y)) \geqq ct\|y_0-y\|^2 - \varepsilon\|f(x_t)-f(u)\|.$$

By the choice of t sufficiently small, we know that

$$\|f(x_t)-f(u)\| \leqq M_0\|x_t-u\| \leqq M_0 t\|y_0-y\|.$$

We have, finally,

$$(f(x_t)-f(u), J(y_0-y)) \geqq \|f(x_t)-f(u)\| \cdot \|y_0-y\|(cM_0^{-1} - \varepsilon\|y_0-y\|^{-1}) \geqq \delta\|f(x_t)-f(u)\| \cdot \|y_0-y\|$$

with $\delta = 2^{-1}cM_0^{-1}$ if we choose $\varepsilon < \delta\|y-y_0\|$. Hence, the hypothesis of Theorem 2 is valid, $B_r(y_1) \subset f(X)$, $f(X)$ is open in Y, and hence $f(X) = Y$. Q.E.D.

PROOF FOR CASE (II). We obtain the desired conclusion by the application of Theorem 3. We choose v in X such that $\|v\| = \|y - y_0\|$, $\phi(v) = J(y_0 - y)$. (Such a choice is possible since $\phi(X) = Y^*$, under the hypothesis of (II).) Let $x_t = u + tv$ where $f(u) = y$, for a given point y of $f(X)$. By the strong ϕ-accretiveness of f, we have

$$t(f(x_t)-f(u), \phi(v)) \geqq ct^2\|y_0-y\|^2,$$

i.e.,

$$(f(x_t)-y, J(y_0-y)) \geqq ct\|y_0-y\|^2.$$

On the other hand,

$$\|f(x_t) - f(u)\|^2 \leqq \varepsilon(t)\|x_t - u\| \leqq \varepsilon(t)t\|y_0 - y\|$$

where $\varepsilon(t) \to 0$ as $t \to 0_+$. Hence, given a constant M, we can choose $t > 0$ so small that

$$M\|f(x_t) - f(u)\|^2 < (f(x_t) - f(u), J(y_0 - y)).$$

Hence, the criterion of Theorem 3 applies and each y_0 lies in $f(X)$. Q.E.D.

We remark that existence results for strongly ϕ-accretive mappings f have been obtained by the writer in his systematic paper [**2**] under the assumption that f is locally Lipschitzian without structural hypotheses upon the Banach space Y and with the additional hypothesis that there exists a surjective strongly ϕ-accretive locally Lipschitzian map R of X on Y. Theorem 4 under the hypothesis of a Holder condition of order $\frac{1}{2}$ is a first step in the program of finding a general theory of ϕ-accretive mappings without assumptions upon f of more than continuity (as in the monotone and accretive cases). Such a theory would provide a much-needed methodological link between the technically disparate theories of monotone and accretive mappings.

1. To obtain results for φ-accretive mappings which are not necessarily strongly ϕ-accretive, we apply the following concept:

DEFINITION 2. *Let X and Y be Banach spaces, ϕ a mapping of X into Y^* which satisfies the conditions of Definition* 1, *R a strongly ϕ-accretive mapping of X into Y. Let f be a ϕ-accretive mapping of X into 2^Y. Then f is said to be hypermaximal ϕ-accretive with respect to R if for each $\varepsilon > 0$ and for the mapping f_ε of X into 2^Y given by*

$$f_\varepsilon(x) = f(x) + R(\varepsilon x),$$

the range of f_ε is all of Y for each $\varepsilon > 0$.

If f is a single-valued ϕ-accretive mapping of X into Y which satisfies the hypotheses (I) or (II) of Theorem 4, and R is one as well, then by Theorem 4, f is hypermaximal ϕ-accretive with respect to R.

THEOREM 5. *Let X and Y be Banach spaces, f a ϕ-accretive mapping of X into Y for a mapping ϕ of X into Y^* which is uniformly continuous on bounded subsets of X. Suppose that there exists a bounded continuous strongly ϕ-accretive mapping R of X into Y such that f is hypermaximal ϕ-accretive with respect to R.*

Then, the closure of $f(X)$ in Y is convex.

Theorem 5 extends results of Minty for maximal monotone mappings in finite-dimensional Banach spaces, results of Rockafellar for maximal monotone maps in reflexive Banach spaces, and a result of Brezis-Pazy for hypermaximal accretive operators from Y to Y with Y^* uniformly convex.

PROOF OF THEOREM 5. Since R and f_ε are strongly ϕ-accretive for each $\varepsilon > 0$, it follows easily that they have single-valued inverses, S and S_ε, from their ranges in Y to X which are Lipschitzian mappings. By assumption, $R(f_\varepsilon) = Y$, so that for each $\varepsilon > 0$, S_ε is a Lipschitzian mapping of Y into X. Let w be an arbitrary element of Y, $u_\varepsilon = S_\varepsilon(w)$. Then there exists a unique element w_ε of $f(u_\varepsilon)$ such that $w = w_\varepsilon + R(\varepsilon u_\varepsilon)$. Hence, $\varepsilon u_\varepsilon = S(w - w_\varepsilon)$. We may assume without loss of generality that $0 \in f(0)$, $R(0) = 0$. Then, we have

$$(w, \phi(u_\varepsilon)) = (w_\varepsilon, \phi(u_\varepsilon)) + (R(\varepsilon u_\varepsilon), \phi(u_\varepsilon)) \geqq (R(n\varepsilon u_\varepsilon), \phi(u_\varepsilon)) \geqq c\varepsilon \|u_\varepsilon\|^2,$$

from which it follows that $\|u_\varepsilon\| \leqq \varepsilon^{-1}c^{-1}\|w\|$. Since R is a bounded mapping, there exists a constant M such that $\|R(\varepsilon u_\varepsilon)\| \leqq M$ for all $\varepsilon > 0$ and a fixed w in Y. Hence, $\|w_\varepsilon\| \leqq M + \|w\|$ for all $\varepsilon > 0$.

Let $y \in f(x)$. By the ϕ-accretiveness of f, it follows that for all $\varepsilon > 0$ $(w_\varepsilon - y, \phi(u_\varepsilon - x)) \geqq 0$. Since $u_\varepsilon = \varepsilon^{-1}S(w - w_\varepsilon)$, we obtain

$$(w_\varepsilon - y, \phi(S(w - w_\varepsilon) - \varepsilon x)) \geqq 0.$$

Since $\|w_\varepsilon\|$ is uniformly bounded, S maps bounded sets into bounded sets, and ϕ is uniformly continuous on bounded subsets of X, it follows immediately that

$$\varliminf_{\varepsilon \to 0} (w_\varepsilon - y, \phi(S(w - w_\varepsilon))) \geqq 0.$$

Let $C = \{ y | y \in Y, \varliminf_{\varepsilon \to 0+} (w_\varepsilon - y, \phi(S(w - w_\varepsilon))) \geqq 0 \text{ for every } w \text{ in } Y\}$. Then C is a closed convex subset of Y which contains $f(x)$ and hence contains $\mathrm{cl}(f(X))$. On the other hand, if y lies in C and we apply the definition of C for $w = y$, we have

$$\varlimsup (w_\varepsilon - y, \phi(S(w_\varepsilon - y))) \leqq 0.$$

Let $x_\varepsilon = S(w_\varepsilon - y)$. Then $R(x_\varepsilon) = w_\varepsilon - y$, and

$$(w_\varepsilon - y, \phi(S(w_\varepsilon - y))) = (R(x_\varepsilon), \phi(x_\varepsilon)) \geqq c\|x_\varepsilon\|^2.$$

Hence x_ε converges to 0 as $\varepsilon \to 0$, and by the continuity of R, w_ε converges to y. Thus $C \subset \mathrm{cl}(f(X))$. Hence $\mathrm{cl}(f(X)) = C$, and $\mathrm{cl}(f(X))$ is convex. Q.E.D.

THEOREM 6. *Let X and Y be Banach spaces, f a ϕ-accretive mapping from X to Y for a map ϕ of X into Y^* which is uniformly continuous on bounded sets with $\phi(X) = Y^*$. Suppose that there exists a bounded continuous mapping R of X into Y which is strongly ϕ-accretive such that f is hypermaximal ϕ-accretive with respect to R. Suppose moreover that $f(X)$ is* *closed in Y and that f is strictly ϕ-accretive.*

Then $f(X) = Y$.

PROOF OF THEOREM 6. By Theorem 5, $\mathrm{cl}(f(X))$ is convex. By hypothesis, $f(X)$ is closed in Y, so that $f(X)$ is a closed convex subset of Y. If $f(X) \neq Y$, then by a simple variant of the Bishop-Phelps Theorem [**1**], there exists a support point of $f(X)$ in Y, i.e. a nonzero y^* in Y^* and an element $y = f(u)$ in $f(X)$ such that $(y^*, y) = \sup_{x \in X}(y^*, f(x))$. We choose v in X such that $\phi(v) = y^*$. Then for $x = u + v$, we have $(y^*, f(x)) = (\phi(v), f(x)) = (\phi(x - u), f(x)) > (\phi(x - u), f(u)) = (y^*, y)$ by the strict ϕ-accretiveness of f. This contradiction proves that $f(X) = Y$.

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