THE RANGE OF m-DISSIPATIVE SETS

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Let $X$ be a real Banach space and $X^*$ its dual space. We shall give some sufficient conditions for an $m$-dissipative set $A$ to have range $R_A$ all of $X$ or to be dense in $X$. The theorems which we shall prove are the following:

THEOREM 1. If $A$ is a coercive, $m$-dissipative set on $X$, then $R_A = X$.

THEOREM 2. In addition to the assumptions of Theorem 1, suppose that there is a compact operator $c$ on $X$ and a strictly increasing right-continuous function $\lambda$ such that

$$
\lambda(0) = 0 \quad \text{and} \quad \lambda(||x_1 - x_2||) \leq ||y_1 - y_2 - (cx_1 - cx_2)||
$$

whenever $[x_1, y_1], [x_2, y_2] \in A$. Then $R_A = X$.

THEOREM 3. Let $X$ be a reflexive Banach space. If $A$ is a coercive, demi-closed, $m$-dissipative set on $X$, then $R_A = X$.

DEFINITION. A mapping $J$ of $X$ into $2^{X^*}$ is said to be the duality mapping if $Jx = \{w \in X^*; ||w|| = ||x||, w(x) = ||x||^2\}$ for all $x \in X$.

It is easy to see that for each $x \in X$, $Jx$ is a nonempty, closed, convex, bounded subset of $X^*$. Thus, for any $z \in X$, $x \in X$, there is $y \in Jx$, such that $y(z) = \inf\{w(z) : w \in Jx\}$ and we use $\langle z, x \rangle$ to denote $y(z)$.

DEFINITION. $A$ is said to be a dissipative set on $X$ if $A$ is a subset of $X \times X$ such that for $[x_1, y_1], [x_2, y_2] \in A$, $\langle y_1 - y_2, x_1 - x_2 \rangle \leq 0$.

T. Kato [5] showed that the above definition is equivalent to the following: for every pair $[x_1, y_1], [x_2, y_2]$ in $A$ and $t \geq 0$,

$$
||x_1 - x_2 - t(y_1 - y_2)|| \geq ||x_1 - x_2||.
$$

Hence, if $A$ is a dissipative set then $(1 - tA)^{-1}$ is a nonexpansive mapping on $R_{(1 - tA)}$ into $X$ for $t \geq 0$. We will say that $A$ is $m$-dissipative if $R_{(1 - tA)} = X$ for all $t \geq 0$. It is known that $A$ is $m$-dissipative if and only if $A$ is dissipative and $R_{(1 - tA)} = X$ (see S. Óharu [6]).

DEFINITION. A dissipative set $A$ is said to be coercive if $A^{-1}(B) = \{y \in X ; Ay \cap B \neq \emptyset\}$ is bounded whenever $B$ is a bounded subset of $X$.

DEFINITION. $A$ is said to be demiclosed if $A$ has the property that $x_n \to x_0$, $y_n \to y_0$, $[x_n, y_n] \in A$, for all $n = 1, 2, \ldots$, implies $[x_0, y_0] \in A$.

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Clearly, if $T$ is accretive in the sense of Browder [1], [2], [3], then $-T$ is dissipative (the converse is not true). We can easily show that $-T$ is $m$-dissipative in Browder’s results [3, Theorem 5 and Theorem 6], and thus, Theorem 3 is a generalization of those results. We note that Theorem 1 is set in a general Banach space and in this case, under the assumptions of Theorem 1, $R_A$ may not equal $X$ (see R. Martin [4]).

**PROOF OF THEOREM 1.** Since $A$ is $m$-dissipative, so is $A^\mu = A - \mu I$ for all $\mu \geq 0$. Thus for any $\mu > 0$, $\eta > 0$ and $y_i \in A\mu x_i$, $i = 1, 2$, then

$$
\|(x_1 - x_2) - \eta(y_1 - y_2)\| = \|(1 + \mu\eta)(x_1 - x_2) - \eta((y_1 + \mu x_1)
-(y_2 + \mu x_2))\| \geq (1 + \mu\eta)\|x_1 - x_2\|.
$$

Hence $(1 + \eta A\mu)^{-1}$ is a Lipschitz continuous mapping on $X$ with Lipschitz constant $(1 + \eta A\mu)^{-1}$ and hence there is $x_\mu \in X$ such that $(1 + \eta A\mu)^{-1}x_\mu = x_\mu$ or $\mu x_\mu \in Ax_\mu$. Now we want to show that $\{\mu x_\mu; 0 < \mu \leq \delta\}$ is bounded. For $0 < \mu \leq \nu \leq \delta$,

$$
\mu\|x_\mu - x_\nu\|^2 \leq \mu\langle x_\mu - x_\nu, x_\mu - x_\nu\rangle - \langle \mu x_\mu - \nu x_\nu, x_\mu - x_\nu\rangle
\leq (\nu - \mu)\|x_\mu\|\|x_\mu - x_\nu\|.
$$

Thus, $\mu\|x_\mu\| \leq \nu\|x_\mu\|$ and we have shown that $\{\mu x_\mu; 0 < \mu \leq \delta\}$ is bounded. It follows from the coercivity of $A$ that $\{x_\mu; 0 < \mu \leq \delta\}$ is bounded, thus $\mu x_\mu \to 0$ as $\mu \to 0$, and $0 \in R_A$. Since, for any $x \in X$, the set $A_1 = \{(\mu, \nu - x); (\mu, \nu) \in A\}$ is coercive and $m$-dissipative, it follows from the above argument that we have $0 \in R_A$, or $x \in R_A$. Consequently, $R_A = X$.

The proof of Theorem 2 follows directly from Theorem 1 and the lemma below. The proof of the lemma is straightforward.

**LEMMA.** Let $A$ be a closed subset of $X \times X$. If there is a compact operator $c$ on $X$ and a strictly increasing right-continuous function $\lambda$ on $[0, \infty)$ such that $\lambda(0) = 0$ and for $[x_1, y_1], [x_2, y_2]$ in $A$, $\|y_1 - y_2 - (cx_1 - cx_2)\| \geq \lambda(\|x_1 - x_2\|)$, then $R_A$ is closed.

**PROOF OF THEOREM 3.** By Theorem 1 we need only to show that $R_A$ is closed. For $y_0 \in R_A$, there is sequence $\{[x_n, y_n]; n = 1, 2, \ldots\}$ in $A$ such that $y_n \to y_0$. Since $\{y_n\}$ is bounded and $A$ is coercive, $\{x_n\} \subseteq A^{-1}(\{y_n\})$ is bounded. By the reflexivity of $X$ we may assume that $x_n \to x_0$ for some $x_0 \in X$. It follows, from the demiclosedness of $A$, $[x_0, y_0] \in A$. Hence, $R_A$ is closed.

**REFERENCES**


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