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THE RANGE OF m-DISSIPATIVE SETS

BY CHI-LIN YEN[1]

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Let X be a real Banach space and X^* its dual space. We shall give some sufficient conditions for an m-dissipative set A to have range R_A all of X or to be dense in X. The theorems which we shall prove are the following:

THEOREM 1. *If A is a coercive, m-dissipative set on X, then $\bar{R}_A = X$.*

THEOREM 2. *In addition to the assumptions of Theorem 1, suppose that there is a compact operator c on X and a strictly increasing right-continuous function λ such that*

$$\lambda(0) = 0 \quad \textit{and} \quad \lambda(\|x_1 - x_2\|) \leqq \|y_1 - y_2 - (cx_1 - cx_2)\|$$

whenever $[x_1, y_1], [x_2, y_2] \in A$. Then $R_A = X$.

THEOREM 3. *Let X be a reflexive Banach space. If A is a coercive, demiclosed, m-dissipative set on X, then $R_A = X$.*

DEFINITION. A mapping J of X into 2^{X^*} is said to be the *duality mapping* if $Jx = \{w \in X^*; \|w\| = \|x\|, w(x) = \|x\|^2\}$ for all $x \in X$.

It is easy to see that for each $x \in X$, Jx is a nonempty, closed, convex, bounded subset of X^*. Thus, for any $z \in X$, $x \in X$, there is $y \in Jx$, such that $y(z) = \inf\{w(z) : w \in Jx\}$ and we use $\langle z, x\rangle$ to denote $y(z)$.

DEFINITION. A is said to be a *dissipative set* on X if A is a subset of $X \times X$ such that for $[x_1, y_1], [x_2, y_2]$ in A, $\langle y_1 - y_2, x_1 - x_2\rangle \leqq 0$.

T. Kato [**5**] showed that the above definition is equivalent to the following: for every pair $[x_1, y_1], [x_2, y_2]$ in A and $t \geqq 0$,

$$\|x_1 - x_2 - t(y_1 - y_2)\| \geqq \|x_1 - x_2\|.$$

Hence, if A is a dissipative set then $(1 - tA)^{-1}$ is a nonexpansive mapping on $R_{(1-tA)}$ into X for $t \geqq 0$. We will say that A is *m-dissipative* if $R_{(1-tA)} = X$ for all $t \geqq 0$. It is known that A is m-dissipative if and only if A is dissipative and $R_{(1-A)} = X$ (see S. Ôharu [**6**]).

DEFINITION. A dissipative set A is said to be *coercive* if $A^{-1}(B) = \{y \in X; Ay \cap B \neq \emptyset\}$ is bounded whenever B is a bounded subset of X.

DEFINITION. A is said to be *demiclosed* if A has the property that $x_n \rightharpoonup x_0$, $y_n \to y_0$, $[x_n, y_n] \in A$, for all $n = 1, 2, \ldots$, implies $[x_0, y_0] \in A$.

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Clearly, if T is accretive in the sense of Browder [**1**], [**2**], [**3**], then $-T$ is dissipative (the converse is not true). We can easily show that $-T$ is m-dissipative in Browder's results [**3**, Theorem 5 and Theorem 6], and thus, Theorem 3 is a generalization of those results. We note that Theorem 1 is set in a general Banach space and in this case, under the assumptions of Theorem 1, R_A may not equal X (see R. Martin [**4**]).

PROOF OF THEOREM 1. Since A is m-dissipative, so is $A_\mu = A - \mu I$ for all $\mu \geqq 0$. Thus for any $\mu > 0$, $\eta > 0$ and $y_i \in A_\mu x_i$, $i = 1, 2$, then

$$\|(x_1 - x_2) - \eta(y_1 - y_2)\| = \|(1 + \mu\eta)(x_1 - x_2) - \eta((y_1 + \mu x_1) - (y_2 + \mu x_2))\| \geqq (1 + \mu\eta)\|x_1 - x_2\|.$$

Hence $(1 + \eta A_\mu)^{-1}$ is a Lipschitz continuous mapping on X with Lipschitz constant $(1 + \eta\mu)^{-1}$ and hence there is $x_\mu \in X$ such that $(1 + \eta A_\mu)^{-1}x_\mu = x_\mu$ or $\mu x_\mu \in Ax_\mu$. Now we want to show that $\{\mu x_\mu; 0 < \mu \leqq \delta\}$ is bounded. For $0 < \mu \leqq \nu \leqq \delta$,

$$\mu\|x_\mu - x_\nu\|^2 \leqq \mu\langle x_\mu - x_\nu, x_\mu - x_\nu\rangle - \langle \mu x_\mu - \nu x_\nu, x_\mu - x_\nu\rangle \leqq (\nu - \mu)\|x_\nu\| \, \|x_\mu - x_\nu\|.$$

Thus, $\mu\|x_\mu\| \leqq \nu\|x_\nu\|$ and we have shown that $\{\mu x_\mu; 0 < \mu \leqq \delta\}$ is bounded. It follows from the coercivity of A that $\{x_\mu; 0 < \mu \leqq \delta\} \subseteq A^{-1}(\{\mu x_\mu; 0 < \mu \leqq \delta\})$ is bounded, thus $\mu x_\mu \to 0$ as $\mu \to 0$, and $0 \in \bar{R}_A$. Since, for any $x \in X$, the set $A_1 = \{(\mu, \nu - x); (\mu, \nu) \in A\}$ is coercive and m-dissipative, it follows from the above argument that we have $0 \in \bar{R}_{A_1}$ or $x \in \bar{R}_A$. Consequently, $\bar{R}_A = X$.

The proof of Theorem 2 follows directly from Theorem 1 and the lemma below. The proof of the lemma is straightforward.

LEMMA. *Let A be a closed subset of $X \times X$. If there is a compact operator c on X and a strictly increasing right-continuous function λ on* $[0, \infty)$ *such that $\lambda(0) = 0$ and for $[x_1, y_1], [x_2, y_2]$ in A, $\|y_1 - y_2 - (cx_1 - cx_2)\| \geqq \lambda(\|x_1 - x_2\|)$, then R_A is closed.*

PROOF OF THEOREM 3. By Theorem 1 we need only to show that R_A is closed. For $y_0 \in \bar{R}_A$, there is sequence $\{[x_n, y_n]; n = 1, 2, \ldots\}$ in A such that $y_n \to y_0$. Since $\{y_n\}$ is bounded and A is coercive, $\{x_n\} \subseteq A^{-1}(\{y_n\})$ is bounded. By the reflexivity of X we may assume that $x_n \rightharpoonup x_0$ for some $x_0 \in X$. It follows, from the demiclosedness of A, $[x_0, y_0] \in A$. Hence, R_A is closed.

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Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37203

Institute of Mathematics, Academia Sinica, Taipei, Taiwan