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# $BV$-FUNCTIONS ON COMMUTATIVE SEMIGROUPS

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The concept of functions of bounded variation on a linearly ordered set has been generalized to a distributive lattice [**1**] and more recently to a semilattice (cf. [**3**], [**4**] and [**5**]). Here, using different techniques, we further extend this notion to commutative semigroups with identity and show that the $BV$-functions characterize the "abstract moment sequences" or what we call moment functions.

Let $S$ be a commutative semigroup with identity 1. A nontrivial homomorphism, which maps $S$ into the multiplicative semigroup of nonnegative real numbers not greater than 1, will be called an *exponential.* We will denote the set of all exponentials on $S$ by $\exp(S)$. Equipped with the topology of simple convergence, $\exp(S)$ is a compact Hausdorff space. We now formulate the *abstract moment problem.* Given a real-valued function $f$ on $S$, when does there exist a regular Borel measure $\mu_f$ on $\exp(S)$ such that $f(x) = \int_{e\in\exp(S)} e(x)d\mu_f(e)$ for all $x \in S$? The Stone-Weierstrass theorem implies the uniqueness of the representing measure (cf. [**2**]), when it exists. Thus using the terminology of [**6**], the abstract moment problem is completely determined. Those functions on $S$ which admit representing measures will be called *moment functions.*

The exponentials of the semigroup $N$ of nonnegative integers under addition can be identified with the closed unit interval $[0, 1]$ in a natural way. Hence, if $S = N$, the abstract moment problem reduces to the already solved little moment problem of Hausdorff and a real-valued function on $N$ is a moment function if and only if it is a moment sequence in the classical sense (cf. [**7**, p. 100]).

Our main result is then that the moment functions and $BV$-functions agree. The methods used provide a new proof of the classical characterization of moment sequences. Principal results on $BV$-functions contained in [**1**], [**3**] and [**4**] also follow in a new way.

Let $f$ be a real-valued function on $S$ and $x \in S$. The translate function $f_x$ of $f$ by $x$ is defined in the usual way by $f_x(y) = f(xy)$ for $y \in S$. Successive differences of $f$ can be defined inductively by

$$\Delta_0 f(\circ) \equiv f(\circ) \quad \text{and} \quad \Delta_n f(\circ; h_1, \ldots, h_n) = \Delta_{n-1}(f - f_{h_n})(\cdot; h_1, \cdot + \cdot, h_{n-1})$$

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where $n \in N$ and $h_i \in S$. Thus $\Delta_n f(\circ; h_1, \ldots, h_n)$ is a function of one variable with increments $h_i$ $(i = 1, 2, \ldots, n)$. The integers $\{0, 1, \ldots, n\}$ will be denoted by $I_n$ and $I_n^k$ will denote the set of functions from $I_k$ into $I_n$, the image of $j \in I_k$ under $i \in I_n^k$ being denoted by $i_j$. We let $X = \{x_j | j \in I_k\}$ be a finite subset of $S$, and set

$$\Delta f(\{x_j\}_j; i, n) = \prod_j \binom{n}{i_j} \Delta f\left(\prod_j (x_j)^{i_j}; \underbrace{x_1, \ldots, x_1}_{n - i_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{n - i_k}\right)$$

where each $x_j$ appears $n - i_j$ successive times as an increment in the difference on the right. The *total variation* $V(f)$ of $f$ is defined by

$$V(f) = \sup_{n,X} \sum_{i \in I_k} |\Delta f(\{x_j\}; i, n)|.$$

The function $f$ is said to be a *BV-function* ($f \in BV(S)$) if $V(f) < \infty$. Our main result is as follows.

THEOREM. *A real-valued function $f$ on $S$ is a moment function if and only if $f \in BV(S)$.*

Details of the proof will appear elsewhere. Below, we only include a brief outline.

Note that the total variation $\|\mu\|$ of a regular Borel measure $\mu$ on a compact Hausdorff space can be expressed as

$$\sup \sum_{p_i \in \mathcal{P}} \left| \int p_i \, d\mu \right|,$$

where the supremum is taken over a sufficiently large collection of partitions of unity. If $\mu_f$ is the representing measure for the moment function $f$ and if we define the continuous real-valued function $\hat{x}$ on $\exp(S)$ by $\hat{x}(e) = e(x)$ for each $x \in S$ then

$$\Delta f(\{x_j\}; i, n) = \int \prod_j \binom{n}{i_j} \hat{x}_j^{i_j} (1 - \hat{x}_j)^{n - i_j} \, d\mu_f.$$

The "only if" part of the assertion now follows from the theory of Bernstein polynomials. For using these facts it can be seen that there are enough partitions of unity of the form

$$\left\{ \prod_j \binom{n}{i_j} \hat{x}_j^{i_j} (1 - \hat{x}_j)^{n - i_j} \middle| i \in I_n^k \right\}$$

to realize the total variation $\|\mu_f\|$ of $\mu_f$ as above.

Conversely, suppose $f \in BV(S)$. Recall that $f$ is said to be *completely monotonic* if each finite difference $\Delta_n f(x; h_1, \ldots, h_n)$ is nonnegative. It follows from [**2**] that any function which is the difference of two completely monotonic functions is a moment function. Through a series of lemmas one can show

$$V(f - f_x) = V(f) - V(f_x)$$

for all $x \in S$. From this it follows that both $f^+$ and $f^-$ as defined by $f^+(x) = \frac{1}{2}[V(f_x) + f(x)]$ and $f^-(x) = \frac{1}{2}[V(f_x) - f(x)]$ are completely monotonic. Thus $f$ is a moment function.

The Banach lattice $M(\exp(S))$ of regular Borel measures on $\exp(S)$ can be identified with $BV(S)$ via the linear map $f \to \mu_f$. Hence $BV(S)$ is a Banach lattice and as such all lattice operations can be described in terms of the variation of its members. We will denote the variation function of $f \in BV(S)$ by $|f|$. That is, $|f| = f \vee (-f)$.

COROLLARY. *If $f \in BV(S)$ then $|f|(x) = V(f_x)$ and $V(f) = \|\mu_f\|$.*

If $S$ is a semilattice, then the above corollary implies the equivalence between our notion and that introduced in [3]. The equivalence between our concept and that of [5] (as well as the classical concept for linearly ordered sets) follows routinely once the latter is formulated in our setting. If $S$ is a distributive lattice and $f$ is a $BV$-valuation, then it follows from [3] that $V(f)$ is the total variation as defined in [1].

Other properties of $BV(S)$ can be easily derived from the results of [2] which we feel should be mentioned here. First of all, the convolution $\mu * \nu$ of two regular Borel measures $\mu$ and $\nu$ on the compact semigroup $\exp(S)$ can be defined in the usual way. The following result easily follows.

PROPOSITION. *If $f$, $g \in BV(S)$ admit representing measures $\mu_f$ and $\mu_g$, respectively, then $\mu_f * \mu_g = \mu_{fg}$. In particular, $BV(S)$ is a Banach algebra under pointwise multiplication.*

A second consequence of [2] is as follows.

PROPOSITION. *The positive cone of the Banach lattice $BV(S)$ is the cone of completely monotonic functions on $S$.*

Consequently, a necessary and sufficient condition that a function be a member of $BV(S)$ is that it is the difference of two completely monotonic functions. Since the completely monotonic functions on a linearly ordered set (regarding this set as a semilattice under $\wedge$) are the nonnegative nondecreasing functions, we get the well-known decomposition of the classical $BV$-functions into monotonic functions.

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