BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, March 1972

ON THE SPECTRUM OF ALGEBRAIC K-THEORY

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Communicated by H. Bass, October 4, 1971

ABSTRACT. The groups $K_i(A)$ of Bass for $i < 0$ are identified as homotopy groups of the spectrum of algebraic K-theory. The spectrum itself is identified. Applications to Laurent polynomials and to K-theory exact sequences are given.

Quillen has recently proposed a K-theory for unital rings [**12**], [**13**]. He associates to a ring A a space $BG1(A)^+$ whose homology is that of the group $G1(A)$ and whose homotopy groups $\pi_i\, BG1(A)^+$ he defines as $K_i(A)$, $i \geqq 1$. The space $BG1(A)^+$ is known to be an H-space, and indeed an infinite loop space.

Hence one is motivated to define $K_i(A)$, for $i \in Z$, as $\pi_i(E(A))$ where $E(A)$ is the associated Ω-spectrum. This note describes $E(A)$ and identifies the groups $K_i(A)$, $i < 0$. In fact, we show that the groups $K_i(A)$ are exactly the groups $L^{-i}K_0(A)$ discussed in Bass' book [**3**, p. 664] for $i < 0$.

Recall from the work of Karoubi and Villamayor [**10**] the cone CA and suspension SA of a ring A. An infinite matrix is called permutant if it is an infinite permutation matrix times a diagonal matrix of finite type. The diagonal matrix is of finite type if its diagonal entries are chosen from a finite subset of the ring. The ring CA is the ring generated by permutant matrices. The cone CA contains the two-sided ideal $\tilde{A} = \bigcup_n M_n(A)$ and the quotient ring is called the suspension of A. We can now state our main result.

THEOREM A. *The space* $\Omega\ (BG1(SA)^+)$ *has the homotopy type of* $K_0(A) \times BG1(A)^+$.

COROLLARY. *For all* $i \in Z$ *we have* $K_i(A) = K_{i+1}(SA)$.

Since Karoubi [**9**] has already identified $K_0(S^iA)$ with Bass' groups $K_{-i}(A)$, the Corollary above completes the identification of Bass' groups with the negative homotopy of the spectrum $E(A)$.

In proving Theorem A we must first analyze the cone construction.

THEOREM B. *The space* $BG1(CA)^+$ *is contractible.*

This result generalizes work of Karoubi and Villamayor [**11**] who show that $K_i(CA) = 0$ for $i \leqq 2$. To prove Theorem B we observe that it suffices

AMS 1970 *subject classifications.* Primary 18F25, 55B15, 16A54, 13D15, 55F50, 18G30, 55B20, 55D35.

Key words and phrases. Algebraic K-theory, suspension and cone of a ring, Ω-spectrum, flasque category, locally free simplicial monoid, group completion functor, Laurent polynomials.

to show that $\tilde{H}_*(BG1(CA)^+) = 0$. This latter result is a consequence of a theorem of Karoubi [**7**] that the category $\mathscr{P}(CA)$ is "flasque," together with a recent result of Barratt and Priddy [**2**] which implies

THEOREM C. *Suppose that M is a locally free simplicial monoid (i.e. M_n is a free monoid for all n). Let $\hat{M}$ denote the group completion of M. If $H_*(M, Z)$ is (graded) commutative, then $\pi_0(\hat{M}) = (\pi_0(\hat{M}))$ and $H_*(\hat{M}, Z) = H_*(M, X) \otimes_{Z\pi_0(M)} Z[\pi_0(\hat{M})]$.*

D. W. Anderson informed me that Quillen has also given a proof of Theorem C.

In order to apply Theorem C, it is necessary to use the description of $BG1(CA)^+$ given by D. W. Anderson [**1**] as the group completion of the morphism complex of the "blown-up" permutative category $\mathscr{P}(CA)$.

Let I be the image of $G1(CA)$ in $G1(SA)$. Since $K_1(CA) = 0$ it follows that $I = E(SA)$, the elementary group of matrices. We have two short exact sequences of groups

$$G1(\tilde{A}) \to G1(CA) \to E(SA),$$

and

$$E(SA) \to G1(SA) \to K_1(SA).$$

Recall Z_∞, the integral completion functor of Bousfield and Kan [**4**], [**5**]. We remark that $Z_\infty BG1(A) \simeq BG1(A)^+$, as was established in [**6**]. The following theorem was communicated to us directly by Bousfield.

THEOREM D (A. K. BOUSFIELD). *Let $F \to E \xrightarrow{\pi} B$ be a fibration of connected spaces and suppose (1) $Z_\infty E$ and $Z_\infty B$ are nilpotent and (2) $\pi_1(B)$ acts nilpotently on each $H_n(F)$. Then the inclusion of $Z_\infty F$ in the fibre of the map $Z_\infty \pi$ is a homotopy equivalence, and moreover $Z_\infty F$ is nilpotent.*

One checks the hypotheses of Theorem D for the fibration

$$BE(SA) \to BG1(SA) \to BK_1(SA).$$

The action of $K_1(SA)$ on $H_*(BE(SA))$ is the limit of inner automorphisms, and hence is trivial. We deduce that we have a fibration

$$Z_\infty BE(SA) \to Z_\infty BG1(SA) \to BK_1(SA)$$

with $Z_\infty BE(SA)$ nilpotent. In fact, $Z_\infty BE(SA)$ is easily seen to be the universal cover of $BG1(SA)^+$.

Consider now the commutative diagram whose rows are fibrations

$$\begin{array}{ccccc} F_0 & \to & Z_\infty BG1(CA) & \to & Z_\infty BE(SA) \\ \downarrow & & \downarrow = & & \downarrow \\ F & \to & Z_\infty BG1(CA) & \to & Z_\infty BG1(SA). \end{array}$$

After examining the exact homotopy sequences, it follows that F_0 is a connected component of F. One considers now the fibration

$$BG1(\tilde{A}) \to BG1(CA) \to BE(SA).$$

Again one checks that the hypotheses of Theorem D are satisfied. Here the action of $E(SA)$ on $H_*(BG1(\tilde{A}))$ is trivial. The idea for this observation is already contained in [**9**] in the proof of Lemma 5.9. It follows that $Z_\infty BG1(\tilde{A}) \simeq F_0$. Next we establish

LEMMA. $Z_\infty BG1(\tilde{A}) \simeq Z_\infty BG1(A)$.

This Lemma generalizes [**11**, Proposition 7.4]. After an application of Theorem B, the proof of Theorem A is quickly completed.

Following Karoubi [**9**] we observe that the Corollary of Theorem A has application to the study of Laurent polynomials.

THEOREM E. *For any unital ring A and $n \in Z$ we have*

$$K_n(A[t, t^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus ?.$$

One considers the pairing

$$K_{n-1}(A) \otimes K_1(Z[t, t^{-1}]) \overset{\cup}{\to} K_n(A[t, t^{-1}])$$

which induces a map $K_{n-1}(A) \to K_n(A[t, t^{-1}])$ by $x \mapsto x \cup [t]$. Karoubi's idea to exhibit an inverse to this map is as follows. He defines a homomorphism $A[t, t^{-1}] \to SA$ by sending $\sum a_i t^i$ to the coset of the matrix in CA:

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} \cdots \\ a_1 & a_0 & a_{-1} \cdots \\ a_2 & a_1 & a_0 \quad \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

This induces the map $K_n(A[t, t^{-1}]) \to K_n(SA)$. By the Corollary to Theorem A, $K_n(SA) = K_{n-1}(A)$ and one checks that the resulting map is a left inverse to the cup product map.

We also have results giving a homotopy theoretic interpretation to the K-theoretic exact sequences of a surjection. These K-theory sequences may be deduced from

THEOREM F. *Let q be a two-sided ideal in the ring A. Let F be the homotopy theoretic fibre of the induced map $BG1(A)^+ \to BG1(A/q)^+$. Then there is a canonical isomorphism $\pi_1 F \cong K_1(A, q)$.*

I now know K-theoretic interpretations for the higher homotopy groups of the fibre F. It would be very interesting to interpret the corresponding fibre in the case of a localization $A \to A_S$.

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