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ON PRÜFER RINGS

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An integral domain D with identity is a *Prüfer domain* if each nonzero finitely generated ideal of D is invertible. Many equivalent forms of this condition are known; for example, see [**2**, Exercise 12, p. 93] or [**4**, Chapter 4]. The concept of a Prüfer domain has been generalized to a commutative ring R with identity (R is a *Prüfer ring* if each nonzero finitely generated regular ideal of R is invertible), and the analogues of conditions equivalent in the domain case have been investigated; see [**3**], [**8**], [**5**]. One question in this vein that has resisted solution is:

If each ideal of R generated by a finite set of regular elements is invertible, is R a Prüfer ring?

In this note, we give an example of a ring R_0 that shows the answer to the preceding question is negative. Such a ring R_0 must, of necessity, fail to have what Marot in [**11**] refers to as *property* (P):

The commutative ring S has *property* (P) if each regular ideal of S is generated by a set of regular elements.

Our ring R_0 will have the property that each ideal of R_0 generated by a set of regular elements is principal. Hence our ring R_0 will show that Marot's assertion in [**11**] that such a ring must have property (P) is false. We proceed to define R_0.

Let $D = K[X, Y]$ be a polynomial ring in two indeterminates over a field K, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be the set of maximal ideals of D not containing Y. We let N be the weak direct sum of the family $\{D/M_\lambda\}_{\lambda\in\Lambda}$ of D-modules, and we define $R_0 = D + N$ to be the idealization of D and N [**12**, p. 2]. The set of zero divisors on N is $\bigcup_\lambda M_\lambda$. We observe that if p is a prime element of D not an associate of Y, then because D is a Hilbert ring [**7**, p. 18], Y is not in some maximal ideal of D containing p, and hence $p \in \bigcup_\lambda M_\lambda$. It follows that the set of regular elements on N is $\{\alpha Y^m | \alpha \in K - \{0\}, m \geqq 0\}$, and since elements of N are nilpotent in R_0,

$$\{\alpha Y^m + n | \alpha \in K - \{0\}, m \geqq 0, n \in N\}$$

is the set of regular elements of R_0.

Because $Y(D/M_\lambda) = D/M_\lambda$ for each λ, $YN = N$ and $Y^m N = N$ for each positive integer m. Hence $\alpha Y^m + n = \alpha Y^m + Y^m n_1 = Y^m(\alpha + n_1)$ for some n_1 in N so that $\alpha Y^m + n$ and Y^m differ by the unit factor $\alpha + n_1$ of

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R_0. Consequently, $\{Y^m R_0\}_{m=1}^{\infty}$ is the set of proper regular ideals of R_0, and each ideal of R_0 generated by a set of regular elements of R_0 is principal, hence invertible. R_0 does not have property (P) because the regular ideal $\{X, Y\}R_0 = \{X, Y\}D + N$ of R_0 is not generated by a set of regular elements. R_0 fails to be a Prüfer ring because the finitely generated regular ideal $\{X, Y\}R_0$ of R_0 is not invertible ($XY \notin \{X^2, Y^2\}R_0 = \{X^2, Y^2\}D + N$).

If R is a commutative ring with identity with total quotient ring T, then an equivalent condition in order that R be a Manis valuation ring[2] is that there is a regular prime ideal P of R such that if $t \in T - R$, then $xt \in R - P$ for some element x in P [**9**]. We remark that our ring R_0 is an integrally closed Manis valuation ring.[3] To prove this, we observe that the total quotient ring T_0 of R_0 is $R_0[1/Y] = D[1/Y] + N$. Since N is a common ideal of R_0 and T_0, and since $D = R_0/N$ is integrally closed in $D[1/Y] = T_0/N$, R_0 is integrally closed. Moreover, if $t \in T_0 - R_0$, then $t = (dY^r + n)/Y^s$ for some $d \in D$, $n \in N$, and some integers r and s with $r < s$. Then $Y^{s-r} \in P_0$ and $Y^{s-r}t = d + n_1$, where $n = Y^s n_1$, so that $Y^{s-r}t \in R_0 - P_0$.

References

1. M. Boisen and M. Larsen, *Prüfer and valuation rings with zero divisors* (to appear).

2. N. Bourbaki, *Elements de mathématique*. Fasc. XXXI. *Algèbre commutative*. Chap. 7: *Diviseurs*, Actualités Sci. Indust., no. 1314, Hermann, Paris, 1965. MR **41** #5339.

3. H. S. Butts and W. W. Smith, *Prüfer rings*, Math. Z. **95** (1967), 196–211. MR **35** #173.

4. R. Gilmer, *Multiplicative ideal theory*, Queen's Series in Pure and Appl. Math., no. 12, Queen's University, Kingston, Ontario, 1968.

5. M. Griffin, *Prüfer rings with zero divisors*, J. Reine Angew. Math. **239/240** (1969), 55–67. MR **41** #188.

6. ———, *Valuation theory and multiplication rings*, Queen's University Preprint no. 1970–37.

7. I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR **40** #7234.

8. M. Larsen, *Equivalent conditions for a ring to be a* P-*ring and a note on flat overrings*, Duke Math. J. **34** (1967), 273–280. MR **35** #174.

9. ———, *Ideal theory in Prüfer rings with zero divisors* (to appear).

10. M. E. Manis, *Valuations on a commutative ring*, Proc. Amer. Math. Soc. **20** (1969), 193–198. MR **38** #2134.

11. J. Marot, *Une généralisation de la notion d'anneau de valuation*, C. R. Acad. Sci. Paris Sér. A-B **268** (1969), A1451–A1454. MR **39** #5564.

12. M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math., no. 13, Interscience, New York, 1962. MR **27** #5790.

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[2] Manis valuation rings arise in connection with the theory of primes developed by Harrison, and with the associated valuation theory introduced by Manis [**10**]. Generally speaking, Manis valuation rings in the theory of Prüfer rings are the counterparts of valuation rings in the theory of Prüfer domains. Our next remark shows that the roles of these two concepts in the different theories do not coincide.

[3] Other examples of Manis valuation rings that are not Prüfer rings have been given by Boisen and Larsen [**1**] and by Griffin [**6**].