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INVARIANT SPLITTINGS IN NONASSOCIATIVE ALGEBRAS: A HOPF APPROACH[1]

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The purpose of this short note is to announce generalizations of known invariant splitting theorems due to Taft [**4**], [**5**], [**6**], [**7**] and Mostow [**1**], which have been obtained by Hopf methods. The approach is an outgrowth of techniques developed by M. Sweedler in order to study algebraic groups from a Hopf point of view, and was motivated by several conversations with him.

0. Let (V, Δ, ε) be a coalgebra over the field k which is equipped with the structure of a unitary associative algebra by means of coalgebra morphisms $m: V \otimes_k V \to V$ and $\mu: k \to V$. $A = (V, \Delta, \varepsilon, m, \mu)$ is then a bialgebra and is a Hopf algebra if $\mathrm{id} \in \mathrm{End}_k(V)$ is invertible in the convolution structure [**2**, p. 71]. We will often confuse A with V.

Recall that A^* has a natural associative algebra structure relative to

$$A^* \otimes_k A^* \hookrightarrow (A \otimes_k A)^* \xrightarrow{\Delta^*} A^*, \qquad k \cong k^* \xrightarrow{\varepsilon^*} A^*.$$

An element $\lambda \in A^*$ is called a (left) integral for A if $a^*\lambda = \langle a^*, 1_A\rangle\lambda$ for all $a^* \in A^*$. If $M \xrightarrow{\psi} M \otimes_k A$ is a right A-comodule, then M carries a (rational) left A^*-module structure via

$$A^* \otimes_k M \to A^* \otimes_k M \otimes_k A \to M \otimes_k A^* \otimes_k A \to M \otimes_k k \cong M$$

[**2**, pp. 33–36, 91–92] and one has the adjoint A^*-module structure on $E = \mathrm{End}_k M$ given in [**3**, p. 332] which is characterized by the relation

$$(a^* \rightharpoonup T)(m) = \sum_{(m)} (a^* \leftharpoonup m_{(1)}) \cdot T(m_{(0)}) \quad \text{for } a^* \in A^*, T \in E \text{ and } m \in M.$$

If A has an integral λ which satisfies $\langle \lambda, 1_A\rangle = 1$, then every rational A^*-module is completely reducible. Conversely, if ${}_{A^*}A$ is a completely reducible rational A^*-module (via the regular right A-comodule structure of A) then A has an integral satisfying the above condition.

1. Let $\mathfrak{N}$ be a nonassociative algebra over k, $\mathfrak{R}$ an ideal in $\mathfrak{N}$ with $\mathfrak{R}\mathfrak{R} = \{0\}$, $\mathfrak{S}$ a subalgebra of $\mathfrak{N}$ with $\mathfrak{N} = \mathfrak{S} \oplus \mathfrak{R}$ (as vector spaces). We have

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THEOREM. *Let A be a commutative Hopf algebra and $\psi: \mathfrak{N} \to \mathfrak{N} \otimes_k A$ a comodule structure map which is multiplicative. Assume further that ${}_{A^*}A$ is completely reducible and that $\mathfrak{R}$ is a subcomodule. Then there is a subalgebra of $\mathfrak{N}$ which is a subcomodule and a vector space complement to $\mathfrak{R}$.*

2. Throughout this section $\mathfrak{N}$ is a nonassociative algebra and a right comodule for a commutative Hopf algebra A where the comodule structure map ψ is multiplicative and ${}_{A^*}A$ is completely reducible. Using the preceding result one easily obtains the following:

THEOREM EA. *If $\mathfrak{N}$ is a finite-dimensional associative algebra which is separable modulo its radical $\mathfrak{R}$, and $\mathfrak{R}$ is an A-subcomodule, then there is a subalgebra of $\mathfrak{N}$ which is a subcomodule and vector space complement to $\mathfrak{R}$.*

THEOREM EL. *If $\mathfrak{N}$ is a finite-dimensional Lie algebra over a field of characteristic* 0, *and* $\mathfrak{R} =$ radical $\mathfrak{N}$ *is a subcomodule, then there is a subalgebra of $\mathfrak{N}$ which is a subcomodule and vector space complement to $\mathfrak{R}$.*

One has similar results for the case of alternative or Jordan algebras.

3. In the notation of §2 we let $\mathfrak{S}$ be a subalgebra subcomodule complement to $\mathfrak{R}$ and $\mathfrak{S}_1$ any separable subalgebra subcomodule of $\mathfrak{N}$. For $\mathfrak{B} \subseteq \mathfrak{N}$ we let $\mathfrak{B}^{A^*} = \{v \in \mathfrak{B} \mid a^* \cdot v = \langle a^*, 1_A \rangle v, \text{ for all } a^* \in A^*\}$. We have

THEOREM UA. *Under the hypothesis of* EA *there is an $x \in \mathfrak{R}^{A^*}$ such that conjugation by $1 + x$ induces a comodule morphism carrying $\mathfrak{S}_1$ into $\mathfrak{S}$.*

THEOREM UL. *Under the hypothesis of* EL, *there is an* $x \in (\mathrm{Nil}\,\mathfrak{N})^{A^*}$ (Nil $\mathfrak{N}$, *the nilradical of* $\mathfrak{N}$) *such that* $\exp(adx)$ *induces a comodule morphism carrying $\mathfrak{S}_1$ to $\mathfrak{S}$.*

One has results similar to those in [7] for the case of alternative or Jordan algebras.

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