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SURFACES IN CONSTANT CURVATURE MANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR FIELD

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I. **Statement of results.** For an (n)-dimensional Riemannian manifold M^n, isometrically immersed in an $(n+k)$-dimensional Riemannian manifold $M_{(c)}^{(n+k)}$ of constant sectional curvature c, let H denote the mean curvature vector field of M^n. H is a section of the normal bundle NM^n of the immersion. When $n = 2$, $k = 1$, and $c = 0$ (a surface immersed in E^3), the requirement $|H| =$ constant is classical constant mean curvature. If $k > 1$, however, the condition $|H| =$ constant is usually too weak to prove reasonable generalizations of the classical theorems for surfaces of constant mean curvature in E^3. We investigate a stronger requirement on H; namely, that H be parallel with respect to the induced connection in the normal bundle (for definitions, see II). Then using an analytic construction first employed by H. Hopf [**2**], we obtain

THEOREM 1. *A compact surface M^2 of genus* 0 *immersed in $M^4(c)$, $c \geqq 0$, upon which H is parallel in the normal bundle, is a sphere of radius* $1/|H|$.

This generalizes a theorem of Hopf, who proved that the only immersed surfaces in E^3 of genus 0 with constant mean curvature are spheres [**2**, Chapter 7, §4]. For complete surfaces in E^4, we prove

THEOREM 2. *A complete surface M^2, immersed in E^4, whose Gauss curvature does not change sign, and for which H is parallel in the normal bundle NM^2, is a minimal surface* ($H \equiv 0$), *a sphere, a right circular cylinder, or a product of circles $S^1(r) \times S^1(\rho)$, where* $|H| = \frac{1}{2}(1/r^2 + 1/\rho^2)^{1/2}$.

This extends a theorem of Klotz and Osserman for complete surfaces of constant scalar mean curvature in E^3 [**5**]. It can also be generalized to immersions into $\overline{M}^4_{(c)}$, $c \geqq 0$. Theorem 2 is proved in two steps. First we prove

THEOREM 3. *A piece of immersed surface M^2 in E^4, satisfying the conditions of Theorem* 2 *with $H \neq 0$, is either a sphere or it is flat* ($K = 0$).

Then we establish the following characterization of flat surfaces in E^4 with parallel mean curvature vector fields:

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THEOREM 4. *A piece of immersed surface M^2 in E^4 with parallel mean curvature vector $H \neq 0$ and $K \equiv 0$ is a piece of $S^1(r) \times S^1(\rho)$: the product of two circles of radius r and ρ with the standard flat immersion. (ρ may be infinite, in which case we have a right circular cylinder.)*

Theorem 2 generalizes to immersions in $S^4(c)$.

Surfaces in E^n which lie minimally in hyperspheres of radius r have the same mean curvature vectors as the hypersphere, and consequently have parallel mean curvature vector fields. Such surfaces are pseudo-umbilic ($\varphi_3 \equiv 0$ in the lemma in II). In this case, Itoh [**3**] has proven a special case of Theorem 2 for immersions in E^4 (see also Chen, [**1**]). For minimal surfaces in S^4, Ruh [**8**] has proven a case of Theorem 1, using methods similar to the basic lemma in II. For a wide variety of examples of complete minimal surfaces in S^3, see Lawson [**6**].

It is possible to show the existence of surfaces in E^n and $S^n(c)$ with parallel H and $\varphi_3 \neq 0$ (i.e. they do not lie minimally in hyperspheres). The method employed uses a theorem due to Szczarba [**9**] on existence of immersions in constant curvature manifolds with codimension $k > 1$.

II. Definitions and Main Lemma.

$\bar{\nabla}$ denotes covariant differentiation on $\bar{M}^{n+k}_{(c)}$, and ∇ denotes covariant differentiation on $M^n \subset \bar{M}^{n+k}$. For X, Y, sections of TM^n, $\nabla_X Y = [\bar{\nabla}_X Y]^T$ where $[\ ]^T$ is projection onto TM^n. $[\ ]^N$ is projection onto NM^n.

DEFINITIONS. $B(X, Y) = [\bar{\nabla}_X Y]^N$. B is the second fundamental form tensor of the immersion. Similarly for N, a section of NM^n, $D_X N = [\bar{\nabla}_X N]^N$. D defines a connection on NM^n. $A(X, N) = [\bar{\nabla}_X N]^T$. A is a tensor: A_p: $TM^n \times NM^n \to TM^n$ is bilinear.

For an orthonormal framing $(e_1 \cdots e_n)$ of TM^n, $H = (1/n)\sum_{i=1}^n B(e_i, e_i)$. This definition of H is independent of the framing. A normal vector field N is said to be parallel in the normal bundle NM^n if $D_X N = 0$ for all X in TM^n. This condition implies $|N| = \text{const}$. Thus our assumption that H is parallel in NM^n includes constant mean curvature. ($|H| = c$.)

The Gauss and Codazzi equations, for X, Y, Z sections in TM^n, are

$$(1)\quad R(X,Y)Z = c\{\langle X,Z\rangle Y - \langle Y,Z\rangle X\} + A(X,B(Y,Z)) - A(Y,B(X,Z)),$$

$$(2)\qquad (\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z),$$

where $(\nabla_X B)(Y,Z) = D_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$ (for a reference for the above definitions and equations, see [**4**, Chapter 7]).

For X, Y in TM^n, N in NM^n, $\tilde{R}(X, Y)N = D_X D_Y N - D_Y D_X N - D_{[XY]} N$ is the curvature tensor for D. For $\tilde{R}$, there is a Gauss-type equation

$$(3)\qquad \tilde{R}(X, Y)N = B(X, A(N, Y)) - B(Y, A(N, X))$$

and an equation, analogous to (2),

$$(4)\qquad (\nabla_X A)(Y, N) = (\nabla_Y A)(X, N).$$

For a unit normal vector e_α at $p \in M^n$, the matrix $(\lambda_{ij}^\alpha) = (B(e_i, e_j) \cdot e_\alpha)$ is the "second fundamental form matrix in the α direction." We specify $H/|H|$ as e_{n+1} when $H \neq 0$. Considering an immersed surface ($n = 2$) given in conformal coordinates (u, v): $ds^2 = E(du^2 + dv^2)$, we associate to it a natural framing

$$(e_1, e_2) = \left(\frac{\partial}{\partial u}\bigg/\sqrt{E}, \frac{\partial}{\partial v}\bigg/\sqrt{E}\right)$$

of the tangent bundle, TM^2.

LEMMA. *For an immersed surface, $M^2 \hookrightarrow \overline{M}_{(c)}^{n+k}$ in conformal coordinates, let $H \neq 0$ and e_α be a unit normal vector field with $e_\alpha \cdot H = 0$:*

(a) *if H is parallel in NM^2, then $\varphi_3 = E\{\frac{1}{2}(\lambda_{11}^3 - \lambda_{22}^3) - i\lambda_{12}^3\}$ is an analytic function of $z = u + iv$;*

(b) *if e_α is parallel in NM^2, then $\varphi_\alpha = E\{\lambda_{11}^\alpha - i\lambda_{12}^\alpha\}$ is an analytic function of z;*

(c) *if $k = 2$ and H is parallel, then e_α is parallel, and both φ_3 and φ_α are analytic;*

(d) *under the conditions of* (a) *and* (b), *either $\varphi_3 \equiv 0$ or $\varphi_\alpha = \kappa\varphi_3$ where κ is a real constant.*

SKETCH OF PROOF. (a) Using equation (4) with $X = \partial/\partial u$, $Y = \partial/\partial v$, $N = H$, and the assumption that H is parallel, the equations

$$(5)\quad (E\lambda_{11}^3)_v - (E\lambda_{12}^3)_u = \tfrac{1}{2}E_v(\lambda_{11}^3 + \lambda_{22}^3), \qquad (E\lambda_{12}^3)_u - (E\lambda_{22}^3)_v = \tfrac{1}{2}E_u(\lambda_{11}^3 + \lambda_{22}^3)$$

are obtained. (5) is in the same form as the Codazzi equations in conformal coordinates for surfaces in E^3, only it is expressed for the distinguished normal $e_3 = H/|H|$. Since $\lambda_{11}^3 + \lambda_{22}^3 = 2|H| =$ constant, (5) reduces to the Cauchy-Riemann equations for φ_3.

(b) Proof follows that of (a), using the fact that $\lambda_{11}^\alpha + \lambda_{22}^\alpha = 0$.

(c) Since NM^2 is 2-dimensional, the assumption that H is parallel forces e_α to be parallel. Then (a) and (b) imply analyticity.

(d) Using equation (3) with

$$X = \frac{\partial}{\partial u_1}\bigg/\sqrt{E}, \qquad Y = \frac{\partial}{\partial u_2}\bigg/\sqrt{E}, \quad \text{and} \quad N = e_3,$$

we obtain, using the fact that e_3 is parallel,

$$(6)\qquad 0 = \left(\sum_{k=1}^{2} \lambda_{k2}^3\lambda_{k1}^\alpha - \lambda_{k1}^3\lambda_{k2}^\alpha\right).$$

Note that (6) implies $\operatorname{Im}(\varphi_\alpha \cdot \bar{\varphi}_3) = 0$. So if $\varphi_3 \not\equiv 0$, $\varphi_\alpha/\varphi_3 = \varphi_\alpha \cdot \bar{\varphi}_3/|\varphi_3|^2$ is real. By (a) and (b), it is also meromorphic, hence constant.

III. **Proof of Theorems (Sketch).** Theorem 1 is proved by constructing an analytic differential θ_3 out of the functions $\varphi_3(z)$ of the lemma: in local coordinates, $\theta_3 = \varphi_3 dz^2$. Since M^2 is of genus 0, θ_3 must be identically zero.

Hence $\varphi_3(z) \equiv 0$. This implies that M^2 is pseudo-umbilic ($\lambda_{11}^3 = \lambda_{22}^3$, $\lambda_{12}^3 = 0$). The function φ_4 associated with $e_4, e_4 \cdot H = 0$ is also zero by a similar argument. Hence M^2 is totally umbilic. This implies that M^2 is immersed as a sphere.

To prove Theorem 3, we can consider on M^2 the quadratic analytic differentials θ_3 and θ_4 given locally by $\varphi_3 dz^2$ and $\varphi_4 dz^2$ (where φ_3, φ_4, and z are as in the lemma). If $K \geqq 0$, M^2 is either compact or parabolic by a theorem of Huber (see [**5**, p. 316]). If it is compact, then either $K \equiv 0$ or M^2 is of genus 0. The genus 0 case is a sphere by Theorem 1.

If $K \leqq 0$, then $|H|^2 - K > |H|^2 > 0$. In a neighborhood of each point, we introduce the new metric $d\tilde{s}^2 = E(|H|^2 - K)^{1/2}(du^2 + dv^2)$. Using the equality

$$|\varphi_3|^2 + |\varphi_4|^2 = E^2(|H|^2 - K)$$

and part (d) of the lemma to show that $\Delta \log(|\varphi_3|^2 + |\varphi_4|^2) = 0$, we establish that $d\tilde{s}^2$ is a flat metric. Therefore, the univeral covering surface $\tilde{M}^2$ of M^2 is conformally the plane. The function $|H|^2 - K$ is easily seen to be superharmonic. Since it is bounded below, it must be constant. Hence K is constant, and must be zero.

Theorem 4 is proved by introducing conformal coordinates (u, v) such that $E \equiv 1$. The lemma is used to show that all λ_{ij}^α are constant. Then a rotation of coordinates puts the immersion into the form

$$(u, v) \to \left(r \cos \frac{u}{r}, r \sin \frac{u}{r}, \rho \cos \frac{v}{\rho}, \rho \sin \frac{v}{\rho} \right).$$

The constants r and ρ are determined from the λ_{ij}^α and $|H|$. This immersion is the standard flat immersion of the plane into E^4 as a product of circles.

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