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2^I IS HOMEOMORPHIC TO THE HILBERT CUBE[1]

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1. **Introduction.** For a compact metric space X, let 2^X be the space of all nonempty closed subsets of X whose topology is induced by the Hausdorff metric. One of the well-known unsolved problems in set-theoretic topology has been to identify the space 2^I (for $I = [0, 1]$) in terms of a more manageable definition. Professor Kuratowski has informed us that the conjecture that 2^I is homeomorphic to the Hilbert cube Q was well known to the Polish topologists in the 1920's. In 1938 in [7] Wojdyslawski specifically asked if $2^I \approx Q$ and, more generally, he asked if $2^X \approx Q$ where X is any nondegenerate Peano space. In this paper we outline our rather lengthy proof that $2^I \approx Q$, announce some generalizations to some other 1-dimensional X, and give some of the technical details.

2. **Preliminaries.** If X is a compact metric space, then the Hausdorff metric D on 2^X can be defined as

$$D(A, B) = \inf\{\varepsilon : A \subset U(B, \varepsilon) \text{ and } B \subset U(A, \varepsilon)\}$$

where, for $C \subset X$, $U(C, \varepsilon)$ is the ε-neighborhood of C in X.

An inverse sequence (X_n, f_n) will have, for $n \geqq 1$, bonding maps $f_n : X_{n+1} \to X_n$ and the inverse limit space will be denoted by $\lim(X_n, f_n)$.

The theory of near-homeomorphisms is very important in this work (see §5). If X and Y are homeomorphic metric spaces, then a map $f: X \to Y$ is a *near-homeomorphism* if for each $\varepsilon > 0$ there is a homeomorphism h from X onto Y such that $d(h, f) < \varepsilon$.

THEOREM 2.1 (*Morton Brown* [**1**, *Theorem* 4, *p.* 482]). *Let* $S = \lim(X_n, f_n)$ *where the* X_n *are all homeomorphic to a compact metric space* X *and each* f_n *is a near-homeomorphism. Then* S *is homeomorphic to* X.

The following corollary was not specifically mentioned in [**1**] but it is an easy corollary of the proof of 2.1.

COROLLARY 2.2. *Furthermore, each projection map* $p_n : S \to X_n$ *is a near-homeomorphism.*

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A map $f: X \to Y$ *stabilizes to a near-homeomorphism* if $f \times \mathrm{id}: X \times Q \to Y \times Q$ is a near-homeomorphism. A space X is a *Q-factor* if $X \times Q \approx Q$. This is equivalent to saying that there exists a space Y such that $X \times Y \approx Q$ since if the latter is true then $Q \approx (X \times Y)^\infty \approx X \times (X \times Y)^\infty \approx X \times Q$.

3. **Outline of proof.** If A is a subset of $I = [0, 1]$, let 2^I_A be the subspace of all elements of 2^I that contain A and let $01 = \{0, 1\}$.

LEMMA 3.1. *2^I is a Hilbert cube if 2^I_{01} is a Hilbert cube.*

PROOF. It is proved in [3] that $2^I \approx CC2^I_{01}$ where CX means the cone over X. (The formula $(A, s, t) \to \{(1 - t)(1 - s)a + t : a \in A\}$ defines a map from $2^I_{01} \times I \times I$ to 2^I producing the same identifications as the coning operations.) O. H. Keller proved in [2] that any infinite-dimensional, compact, convex subspace of Hilbert space is homeomorphic to the Hilbert cube. Since CQ has a convenient geometric realization as a convex subset of Hilbert space, we have $CQ \approx Q$, and thus $CCQ \approx Q$ and the result follows.

We now represent 2^I_{01} by two inverse limits, using the first to analyze the second. For each n, let $F_n: 2^I_{01} \to 2^I_{01}$ be the map sending each A to its closed $1/n$-neighborhood in I and let $B_n = F_n(2^I_{01})$. Define $f_n: B_{n+1} \to B_n$ by $f_n = F_{n(n+1)}|B_{n+1}$.

LEMMA 3.2. *$2^I_{01} \approx \lim(B_n, f_n)$.*

PROOF. Since $1/n = 1/(n + 1) + 1/n(n + 1)$, we have $f_n F_{n+1} = F_n$ and thus we can define the map $F: 2^I_{01} \to \lim(B_n, f_n)$ by $F(A) = (F_1(A), F_2(A), \cdots)$, and this is a homeomorphism since the map $(A_1, A_2, \cdots) \to \bigcap_{n=1}^\infty A_n$ is the inverse of F.

THEOREM 3.3. *2^I_{01} is a Q-factor.*

OUTLINE OF PROOF. We show that each B_n is a Q-factor and then establish that each f_n stabilizes to a near-homeomorphism and hence $2^I_{01} \times Q \approx \lim(B_n \times Q, f_n \times \mathrm{id}) \approx Q$ by Theorem 2.1.

For our second inverse system we define the spaces as follows. Let $\sigma(n) = \{0, 1, 1/n, 1/n + 1, \cdots\}$ and let $Y_n = 2^I_{\sigma(n)}$.

COROLLARY 3.4. *$Y_n \approx Q$.*

PROOF. Let $J(n, 1), J(n, 2), \cdots$ denote the subintervals of I determined by $\sigma(n)$ and enumerated from right to left and let $K(n, m)$ be the set of endpoints of $J(n, m)$. The function $\varphi: Y_n \to \prod_{m=1}^\infty 2^{J(n,m)}_{K(n,m)}$ given by $\varphi(A) = (A \cap J(n, 1), A \cap J(n, 2), \cdots)$ is a homeomorphism. Since each $2^{J(n,m)}_{K(n,m)} \approx 2^I_{01}$ and since by [4] any countable infinite product of nondegenerate Q-factors is homeomorphic to Q, we are done.

We define the bonding maps $r_n: Y_{n+1} \to Y_n$ as follows. For each $A \in Y_{n+1}$ let $\delta_n(A)$ be the distance from $1/n$ to the nearest point of A. We define $r_n(A)$

to be the union of A with the two closed intervals of length $\delta_n(A)$ extending towards $1/n$ from the points of A nearest to $1/n$ on either side.

LEMMA 3.5. $2^I_{01} \approx \lim(Y_n, r_n)$.

PROOF. The following observation is useful. If $A \in Y_m$ $(m > n \geqq 1)$, then $r_n^m = r_n \circ \cdots \circ r_{m-1}: Y_m \to Y_n$ acts independently on the closures $\bar{U} = [u, v]$ of the components of $I\backslash A$, with

$$r_n^m(A) \cap \bar{U} = [u, u + \xi_U^n] \cup [v - \xi_U^n, v]$$

where ξ_U^n is the maximum distance from points of $\sigma(n)$ to the complement of U. Define $R_n: 2^I_{01} \to Y_n$ by setting $R_n(A)$ to be the union of A and

$$\bigcup\{[u, u + \xi_U^n] \cup [v - \xi_U^n, v] : U = (u, v) \text{ is a component of } I\backslash A\}.$$

It easily follows that $R_n = r_n \circ R_{n+1}$ by observing what happens, for $A \in 2^I_{01}$, on the component of $I\backslash A$ that contains $1/n$, if it exists. Thus we can define $R: 2^I_{01} \to \lim(Y_n, r_n)$ by $R(A) = (R_1(A), R_2(A), \cdots)$ and this is a homeomorphism since the map $(A_1, A_2, \cdots) \to \bigcap_{n=1}^{\infty} A_n$ is the inverse of R.

THEOREM 3.6. *2^I is a Hilbert cube.*

OUTLINE OF PROOF. By 2.1, 3.1, 3.4, and 3.5 it is sufficient to prove that each r_n is a near-homeomorphism. We proceed as follows. In the representation of Y_n as an infinite product of copies of 2^I_{01}, the map r_n becomes the stabilization of a map ρ_n from $2^{J(n+1,1)}_{K(n+1,1)}$ to $2^{J(n,1)}_{K(n,1)} \times 2^{J(n,2)}_{K(n,2)} \approx 2^{J(n+1,1)}_{1/n+1,1/n,1}$. By normalizing the length of $J(n + 1, 1)$, we regard ρ_n as a map from 2^I_{01} to $2^I_{0,t_n,1}$ for some $t_n \in (0, 1)$. Define $h_m: B_m \to C_m = F_m((2^I_{0,t_n,1}))$ by $h_m(F_m(A)) = F_m(\rho_n(A))$ for $A \in 2^I_{01}$. Then h_m is well-defined (this can be seen by letting $\sigma_m: B_m \to 2^I_{01}$ be the natural [discontinuous] section of F_m and observing that $h_m = F_m \rho_n \sigma_m$) and continuous, $h_m f_m = (f_m|C_{m+1})h_{m+1}$, and hence $\rho_n = \lim(h_m)$.

$$\begin{array}{ccccc} \cdots \leftarrow B_m & \overset{f_m}{\leftarrow} & B_{m+1} & \leftarrow \cdots & 2^I_{01} \\ \downarrow h_m & & \downarrow h_{m+1} & & \downarrow \rho_n \\ \cdots \leftarrow C_m & \overset{f_m}{\leftarrow} & C_{m+1} & \leftarrow \cdots & 2^I_{0,t_n,1} \end{array}$$

(In fact each B_m and C_m is a compact polyhedron and h_m is a deformation retraction.) We show that each h_m stabilizes to a near-homeomorphism and since each f_m does we know by Theorem 5.2 of this paper that ρ_n stabilizes to a near-homeomorphism and hence that r_n is a near-homeomorphism.

4. **Extension to connected graphs and dendrons.** Using the facts that (1) the *collapse-to-base of the mapping cylinder of a map between two Q-factors stabilizes to a near-homeomorphism* [5] *and* (2) *the compactification of a Hilbert cube manifold into a Q-factor by the addition of a Q-factor with*

property Z in the compactification yields a Hilbert cube [6], one may derive the next result from Theorem 3.6.

THEOREM 4.1. *If X is a nondegenerate, connected graph or dendron, then 2^X is a Hilbert cube.*

5. **Near-homeomorphisms and inverse limits.** The material in this section is referred to in the proof of Theorem 3.6.

LEMMA 5.1. *Let X, Y, and Z be compact metric spaces. If $f:X \to Y$ and $h:Y \to Z$ are maps where f and $g = h \circ f$ are near-homeomorphisms, then h is a near-homeomorphism.*

PROOF. Let $\varepsilon > 0$ and pick $\delta > 0$ such that if $y, y' \in Y$ and $d(y, y') < \delta$, then $d(h(y), h(y')) < \varepsilon/2$. Select homeomorphisms $f_1 : X \to Y$ and $g_1 : X \to Z$ with $d(f_1, f) < \delta$ and $d(g_1, g) < \varepsilon/2$. Then $d(f_1 f_1^{-1}, ff_1^{-1}) < \delta$ so $d(hf_1 f_1^{-1}, hff_1^{-1}) = d(h, gf_1^{-1}) < \varepsilon/2$. Since $d(gf_1^{-1}, g_1 f_1^{-1}) < \varepsilon/2$, we have $d(h, g_1 f_1^{-1}) < \varepsilon$ and since $g_1 f_1^{-1}$ is a homeomorphism, then h is a near-homeomorphism.

THEOREM 5.2. *Let $S = \lim(X_n, f_n)$ and $T = \lim(Y_n, g_n)$ where all the spaces are compact and for each n, let $h_n : X_n \to Y_n$ be a map such that $g_n h_{n+1} = h_n f_n$. If for each n, both f_n and h_n are (stabilize to) near-homeomorphisms, then $h = \lim(h_n) : S \to T$ is a (stable) near-homeomorphism.*

PROOF. The stable version of the theorem follows directly from the other by stabilizing the whole system of spaces and maps and noting that in general, $\lim(X_n, f_n) \times Q \approx \lim(X_n \times Q, f_n \times \mathrm{id})$. Otherwise, let $\varepsilon > 0$ and let $n \geqq 1$ be large enough so that the projection map $p_n : T \to Y_n$ is an ε-map. By 5.1, each g_i is a near-homeomorphism, so by 2.2 all projection maps $p_i' : S \to X_i$ and $p_i : T \to Y_i$ are near-homeomorphisms. Since the composition of two near-homeomorphisms is a near-homeomorphism, then $f = h_n p_n' : S \to Y_n$ is a near-homeomorphism. Since $g = p_n : T \to Y_n$ is an ε-map, there exists $\delta > 0$ such that if $t, t' \in T$ where $d(g(t), g(t')) < \delta$, then $d(t, t') < \varepsilon$. Let $\alpha : S \to Y_n$ and $\beta : T \to Y$ be homeomorphisms within $\delta/2$ of f and g, respectively. Thus $d(f, \alpha) < \delta/2$ and $d(\alpha, g\beta^{-1}\alpha) = d(\beta\beta^{-1}\alpha, g\beta^{-1}\alpha) < \delta/2$ and hence $d(f, g\beta^{-1}\alpha) = d(gh, g\beta^{-1}\alpha) < \delta$ which implies that $d(h, \beta^{-1}\alpha) < \varepsilon$. This finishes the proof since $\beta^{-1}\alpha$ is a homeomorphism.

REFERENCES

1. M. Brown, *Some applications of an approximation theorem for inverse limits*, Proc. Amer. Math. Soc. **11** (1960), 478–483. MR **22** #5959.

2. O. H. Keller, *Die Homeomorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.

3. R. M. Schori, *Hyperspaces and symmetric products of topological spaces*, Fund. Math. **68** (1966), 77–88.

4. J. E. West, *Infinite products which are Hilbert cubes*, Trans. Amer. Math. Soc. **150** (1970), 1–25.

5. ———, *Mapping cylinders of Hilbert cube factors*, General Topology **1** (1971), 111–125.

6. ———, *The subcontinua of a dendron form a Hilbert cube factor*, Proc. Amer. Math. Soc. (to appear).

7. M. Wojdyslawski, *Sur la contractilité des hyperspaces de continus localement connexes*, Fund. Math. **30** (1938), 247–252.

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