BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

A GENERALIZATION OF THE HELLY SELECTION THEOREM

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Communicated by Victor Klee, November 16, 1971

1. **Introduction.** In this paper we consider sequences $\{y_k\}$ of real valued functions defined on an interval I. We are interested in finding conditions which when satisfied by the sequence $\{y_k\}$ guarantee the existence of a subsequence of $\{y_k\}$ which converges pointwise on I. With this in mind we make the following definition.

DEFINITION 1.1. Let $f:I \to R$ and consider the set $\mathscr{P}$ of all finite nonempty partitions $P = \{x_1, x_2, \ldots, x_n\}$ of I where $n \geqq 1$ and $x_1 < x_2 < \cdots < x_n$. We denote by $T(f)$ the oscillation of f on I and define it by

$$T(f) = \sup_{P\in\mathscr{P}}\left\{\sum_{i=1}^{n} |f(x_i)| : (-1)^i f(x_i) > 0 \ \forall i \right.$$
$$\left. \text{or } (-1)^i f(x_i) < 0 \ \forall i \text{ or } (-1)^i f(x_i) = 0 \ \forall i \right\}.$$

For a function f which is nonnegative on I the oscillation of f on I, $T(f)$, is the supremum of f on I. It is not the case, however, that the set of f for which $T(f)$ is finite forms a Banach space with norm $T(f)$ since closure under addition is not satisfied. It is also not the case that the set of f for which $T(f)$ is finite forms a metric space with metric given by $d(f, g) = T(f - g)$ because the triangle inequality is not satisfied.

Our main result, for which we give a number of applications later, is the following.

THEOREM 1.2. *Let $\{y_k\}$ be such that $y_k : I \to R$. If $T(y_k - y_j) \leqq M$ for all k, j then $\{y_k\}$ contains a subsequence which converges pointwise on I.*

The original motivation for this theorem comes from the study of boundary value problems. In [3] the author showed, among other things, that if $\{y_k\}$ is a uniformly bounded sequence of continuous real valued functions defined on an interval I having the property that there exists a positive integer N such that y_k and y_j are not equal at more than N values of x for $k \neq j$ then y_k contains a subsequence which converges at every point in I. This result is a corollary of Theorem 1.2. A more complete description of the connection between such convergence theorems and the

AMS 1969 *subject classifications.* Primary 4020; Secondary 3436.
Key words and phrases. Sequences of functions, pointwise convergence.

study of boundary value problems may be found in [3] along with a list of references.

2. **Preliminary results.** We begin by stating a known result due to Ramsey which we will refer to repeatedly. This result and its proof may be found in [**1**, Theorem A] or [**2**, Theorem A, p. 82].

THEOREM 2.1. *Let Γ be an infinite class, u and r positive integers; and let all those subclasses of Γ which have exactly r members, or, as we may say, let all r-combinations of the members of Γ be divided in any manner into u mutually exclusive classes C_i $(i = 1, 2, \ldots, u)$, so that every r-combination is a member of one and only one C_i. Then, assuming the Axiom of Selections, Γ must contain an infinite subclass Δ such that all the r-combinations of the members of Δ belong to the same C_i.*

COROLLARY 2.2. *If $\{f_k\}$ is any sequence of real valued functions defined on an interval J then either there is a subsequence $\{h_j\}$ of $\{f_k\}$ such that $\{h_j\}$ is a monotone sequence on J or else there is a subsequence $\{h_j\}$ of $\{f_k\}$ such that if $i \neq j$ there are $t, \tau \in J$ depending on i, j with $h_i(t) > h_j(t)$ and $h_i(\tau) < h_j(\tau)$.*

PROOF. If there are only finitely many functions in $\{f_k\}$ which are distinct on J then infinitely many are identical on J and we are done. Thus we may assume there are infinitely many functions in $\{f_k\}$ which are distinct on J and, by picking a subsequence if necessary, we may assume all the f_k are distinct. Let $u = r = 2$ and $\Gamma = \{f_k\}$. Let $C_1 = \{\{f_k, f_j\} : k \neq j,\ f_k(t) \leqq f_j(t)\ \ \forall t \in J \text{ or } f_k(t) \geqq f_j(t)\ \ \forall t \in J\}$ and $C_2 = \{\{f_k, f_j\} : k \neq j,\ f_k(t) > f_j(t)$ for some $t \in J$ and $f_k(\tau) < f_j(\tau)$ for some $\tau \in J\}$. The result now follows from Theorem 2.1.

COROLLARY 2.3. *If $\{f_k\}$ is any sequence of real valued functions defined on an interval J and $\varepsilon > 0$ is any number then either there is a subsequence $\{h_j\}$ of $\{f_k\}$ such that, for $i \neq j$, $|h_j(t) - h_i(t)| < \varepsilon$ for all $t \in J$ or else there is a subsequence $\{h_j\}$ of $\{f_k\}$ such that, for $i \neq j$, $|h_j(t) - h_i(t)| \geqq \varepsilon$ for some $t \in J$ depending on i, j.*

PROOF. As in the proof of Corollary 2.2 we may assume the f_k are all distinct on J or we are done. Let $\Gamma = \{f_k\}$, $u = r = 2$, $C_1 = \{\{f_k, f_j\}: k \neq j,\ |f_k(t) - f_j(t)| < \varepsilon\ \ \forall t \in J\}$, $C_2 = \{\{f_k, f_j\}: k \neq j,\ |f_k(t) - f_j(t)| \geqq \varepsilon$ for some $t \in J\}$. The result now follows from Theorem 2.1.

3. **Proof of Theorem 1.2.** For $x \in I$ we have $|y_k(x)| \leqq |y_k(x) - y_1(x)| + |y_1(x)| \leqq M + |y_1(x)|$ so the sequence $\{y_k\}$ is pointwise bounded and hence any monotone subsequence of $\{y_k\}$ is convergent.

Let $\{J_i\}$ be an enumeration of all the nonempty open intervals contained in I with rational endpoints.

It follows from Corollary 2.2 applied to J_1 that either there is a subsequence of $\{y_k\}$, again denoted by $\{y_k\}$, which is monotone and convergent on J_1 or else there is a subsequence of $\{y_k\}$, again denoted by $\{y_k\}$, such that, for $k \neq j$, $y_k(t) > y_j(t)$ and $y_k(\tau) < y_j(\tau)$ for some $t, \tau \in J_1$. We now repeat the process described in the previous sentence consecutively on the intervals $J_2, J_3, \ldots$ and then take the diagonal subsequence, denoted by $\{y_k\}$ again. This sequence has the property that on any of the intervals J_i it is either monotone on J_i for k sufficiently large and hence converges on J_i or else for every $k \neq j$ sufficiently large there are $t, \tau \in J_i$ such that $y_k(t) > y_j(t)$ and $y_k(\tau) < y_j(\tau)$.

Now using J_1 and $\varepsilon = 1$ it follows from Corollary 2.3 that either there is a subsequence of $\{y_k\}$, again denoted by $\{y_k\}$, such that, for $k \neq j$, $|y_k(t) - y_j(t)| < \varepsilon = 1$ for all $t \in J_1$ or else there is a subsequence of $\{y_k\}$, again denoted by $\{y_k\}$, such that if $k \neq j$ there is some $t \in J_1$ for which $|y_k(t) - y_j(t)| \geqq \varepsilon = 1$. We now repeat this process consecutively on J_1 using $\varepsilon = 1/2, \varepsilon = 1/3, \ldots, \varepsilon = 1/n, \ldots$ and then take the diagonal subsequence, again denoted by $\{y_k\}$. This sequence has the property that for J_1 and $\varepsilon = 1/n$ either for all $k \neq j$ sufficiently large $|y_k(t) - y_j(t)| < \varepsilon = 1/n$ holds for all $t \in J_1$ or else for all $k \neq j$ sufficiently large there is some $t \in J_1$ with $|y_k(t) - y_j(t)| \geqq \varepsilon = 1/n$.

We now repeat the entire process described in the preceding paragraph consecutively on the intervals $J_2, J_3, \ldots$ and then take the diagonal subsequence, again denoted by $\{y_k\}$. This sequence has the property that for J_i and $\varepsilon = 1/n$ then either for every $k \neq j$ sufficiently large, depending on i and n, $|y_k(t) - y_j(t)| < \varepsilon = 1/n \quad \forall t \in J_i$ or else for every $k \neq j$ sufficiently large, depending on i and n, $|y_k(t) - y_j(t)| \geqq \varepsilon = 1/n$ for some $t \in J_i$, depending on k, j.

We will now show using proof by contradiction that $\{y_k\}$ converges at all but countably many values of $x \in I$. For $x \in I$ such that $\{y_k(x)\}$ is not convergent let $\{F_{xi}\}$ be the subsequence of $\{J_i\}$ consisting of the intervals which contain x. There must be a smallest positive integer n_{xi} such that $|y_k(t) - y_j(t)| \geqq \varepsilon = 1/n_{xi}$ for all $k \neq j$ sufficiently large for some $t \in F_{xi}$ or else $\{y_k\}$ is Cauchy on F_{xi} and hence converges at x which would contradict the choice of x. If $\overline{\lim}_{i \to +\infty} n_{xi} = +\infty$ then there is a subsequence $n_{xi(\alpha)}$ of $\{n_{xi}\}$ such that $\lim_{\alpha \to +\infty} n_{xi(\alpha)} = +\infty$ and by the definition of $n_{xi(\alpha)}$ we have $|y_k(t) - y_j(t)| < 1/(n_{xi(\alpha)} - 1)$ for all $k \neq j$ sufficiently large and all $t \in F_{xi(\alpha)}$. This implies $\{y_k(x)\}$ is Cauchy which contradicts the choice of x, so $\overline{\lim}_{i \to +\infty} n_{xi} = c_x < +\infty$. Let $d_x > c_x$ be an upper bound for the set $\{n_{xi}\}$.

If there are uncountably many x at which $y_k(x)$ is not convergent then there is some number d so that $d_x \leqq d$ holds for uncountably many x at which $\{y_k(x)\}$ does not converge. Denote this uncountable set of x's by A.

Choose N so that $N(1/d) > M$. Now choose $u(1) \in A \cap I^0$ and

$F_{u(1)i(1)} \in \{F_{u(1)i}\}$ such that $(I - F_{u(1)i(1)}) \cap A$ is uncountable. Choose $u(2) \in (I - F_{u(1)i(1)}) \cap (A \cap I^0)$ and $F_{u(2)i(2)} \in \{F_{u(2)i}\}$ such that $F_{u(1)i(1)} \cap F_{u(2)i(2)} = \emptyset$ and $(I - (F_{u(1)i(1)} \cup F_{u(2)i(2)})) \cap A$ is uncountable. Continuing in this way we get $\{u(1), u(2), \ldots, u(2N + 1)\}$ in $A \cap I^0$ and $\{F_{u(1)i(1)}, F_{u(2)i(2)}, \ldots, F_{u(2N+1)i(2N+1)}\}$ which are mutually disjoint. By renaming the points $u(i)$ we may assume $u(1) < u(2) < \cdots < u(2N + 1)$. Now we choose $k \neq j$ fixed but sufficiently large that for each odd positive integer α, $1 \leqq \alpha \leqq 2N + 1$, $|y_k(x) - y_j(x)| \geqq \varepsilon = 1/n_{u(\alpha)i(\alpha)}$ holds for some $x(\alpha) \in F_{u(\alpha)i(\alpha)}$ and for each even positive integer α, $2 \leqq \alpha \leqq 2N$, $y_k(t_\alpha) > y_j(t_\alpha)$ holds for some $t_\alpha \in F_{u(\alpha)i(\alpha)}$ and $y_k(\tau_\alpha) < y_j(\tau_\alpha)$ holds for some $\tau_\alpha \in F_{u(\alpha)i(\alpha)}$. Consider the partition $P_0 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ where $\beta_\alpha = x(\alpha)$ if α is odd; β_α is omitted from P_0 if α is even and $y_k(x(\alpha - 1)) - y_j(x(\alpha - 1))$ and $y_k(x(\alpha + 1)) - y_j(x(\alpha + 1))$ have opposite signs; β_α is taken to be t_α if $y_k(x(\alpha - 1)) - y_j(x(\alpha - 1)) < 0$ and $y_k(x(\alpha + 1)) - y_j(x(\alpha + 1)) < 0$; β_α is taken to be τ_α if $y_k(x(\alpha - 1)) - y_j(x(\alpha - 1)) > 0$ and $y_k(x(\alpha + 1)) - y_j(x(\alpha + 1)) > 0$. We observe this implies that $T(y_k - y_j) \geqq N(1/d) > M$ for this fixed $k \neq j$ which contradicts the hypotheses of the theorem.

We conclude that $\{y_k\}$ converges for all but countably many values of x and hence a subsequence can be chosen which converges at every point.

COROLLARY 3.1. *Let $\{y_k\}$ be a sequence of real valued functions defined on a nonempty set $S \subset R$. If $T(y_k - y_j) \leqq M$ for all k, j (where partition points are taken in S instead of I) then $\{y_k\}$ contains a subsequence which converges pointwise on S.*

PROOF. Define $\{z_k\}$, $z_k : R \to R$ by

$$\begin{aligned} z_k(x) &= y_k(x), \qquad x \in S, \\ &= 0, \qquad x \notin S, \end{aligned}$$

then $T(z_k - z_j) = T(y_k - y_j) \leqq M$ so Theorem 1.2 applies.

COROLLARY 3.2 (*Helly Selection Theorem*). *Let $\{y_k\}$ be a sequence of functions of bounded variation with $y_k : [a, b] \to R$ and let there exist a number K such that $|y_k(x)| \leqq K$ for $k = 1, 2, \ldots$ and $x \in [a, b]$ and such that $V_a^b(y_k) \leqq K$ for $k = 1, 2, \ldots$ where $V_a^b(y_k)$ is the variation of $\{y_k\}$ on $[a, b]$. Then there exists a subsequence of $\{y_k\}$ which converges at every point.*

PROOF. $T(y_k - y_j) \leqq 2K + V_a^b(y_k - y_j) \leqq 2K + V_a^b(y_k) + V_a^b(y_j) \leqq 4K$. In fact, the limit function is of bounded variation but we are not concerned with that here.

COROLLARY 3.3. *Let $\{y_k\} \in C(I)$ and $D_{kj} = \{x : x \in I, |y_k(x) - y_j(x)| > 0\}$. Then we know that $D_{kj} = \bigcup_{n=1}^{+\infty} I_{kjn}$ where each I_{kjn} is a relatively open interval in I and $I_{kjn} \cap I_{kjm} = \emptyset$ for $n \neq m$. If $\sum_{n=1}^{+\infty} \sup_{x \in I_{kjn}} |y_k(x) - y_j(x)|$*

$\leqq M$ *for all* k, j *then* $\{y_k\}$ *contains a subsequence which converges at each point of* I.

COROLLARY 3.4. *Let* $y_k \in C(I)$ *and* K *a fixed positive integer such that* $\{x : x \in I, |y_k(x) - y_j(x)| > 0\}$ *contains no more than* K *components for each* k, j. *If* $|y_k(x)| \leqq H$ *holds, for* $x \in I$ *and* $k = 1, 2, \ldots$, *then* $\{y_k\}$ *contains a subsequence which converges at every point of* I.

PROOF. $T(y_k - y_j) \leqq KH$.

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