COTERMINAL FAMILIES AND THE STRONG MARKOV PROPERTY

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1. Let $\Omega$ be the set of right continuous paths mapping $[0, \infty)$ into $(E, \rho)$, where $E$ is a locally compact, separable metric space with metric $\rho$ and $E_\Delta = E \cup \{\Delta\}$, $\{\Delta\}$ the point at infinity. Following the notation of [1], we let $X = (\Omega, X, F, F_t, \theta_t, P^x)$ be a strong Markov process with stationary transition probabilities on $(E, \rho)$, and we use $\mathcal{B}$ and $\mathcal{B}_\Delta$ to denote the $\sigma$-fields of Borel sets on $E$ and $E_\Delta$ respectively. The $\sigma$-fields $\{F_t, 0 \leq t\}$ are assumed to be right continuous. Finally we assume a fixed initial distribution $\mu$, and all a.s. statements will refer to $P = \int_{E_\Delta} \mu(dx)P^x$.

The purpose of this note is to outline a type of strong Markov property for a class of random times, resembling last exit times from sets. To motivate the problem, write the strong Markov property at a stopping time $T$ as

$$P(X_t \in A | \mathcal{F}_T) = P^{x_T}(X_{t-T} \in A) \quad \text{a.s.}$$

on $\{T < t\}$, where $\mathcal{F}_T$ is the usual $\sigma$-field of information up through $T$. We are interested in finding an analogue of (1) with $T$ replaced by random times such as $L'$, the last exit before $(t + 0)$ from a given set, and $\mathcal{F}_T$ replaced by an appropriate $\sigma$-field. Since the conditioning now involves the future of the process, the distributions on the right of (1) will be altered but will be seen to depend only on $X_{L'}$ or $X_{L'-}$ and $t - L'$.

As an example suppose $X$ is reflecting Brownian motion on $[0, \infty)$, $P = P^0$ and $L'$ is the last hit of $\{0\}$ prior to $t$. We leave it to the reader to verify that if $\sigma(L')$ is the $\sigma$-field generated by $L'$, then

$$P^0(X_t \in A | \sigma(L')) = \int_A \frac{y \exp(-y^2/2(t - L'))}{t - L'} \, dy \quad \text{a.s.}$$

on $\{0 < L' < t\}$. Moreover another computation shows that

$$\int_0^\infty \frac{1}{\sqrt{s}} \left( \frac{y \exp(-y^2/2s)}{s} \right) \, dy \right) P^0(T_{\{0\}} > t, X_t \in A)$$

(3) $=$ $\int_A \frac{1}{\sqrt{s + t}} \left( \frac{x \exp(-x^2/2(t + s))}{s + t} \right) \, dx$.
i.e., the new distributions can be normalized so that they form an entrance law relative to the family of transition probabilities

\[ H_t(x, A) = P^x\{X_t \in A, \, T(0) > t \}. \]

The results we summarize here in essence generalize (2) and (3) to a wide class of strong Markov processes, including standard processes, and to arbitrary coterminal families \( \{L^t, t \geq 0\} \) as defined below. Two different \( \sigma \)-fields will be used in the analogues of (2), and the resulting families of probability distributions will depend only on \( X_{L^t} \) (or \( X_{L^t-} \)) and \( t - L^t \). In both cases a suitable normalization renders the families entrance laws relative to an appropriate transition probability. Details and proofs of these results will appear elsewhere.

2. Following [3] we characterize our random times axiomatically. Let \( k_t \) be the killing operator defined by \((k_t \omega)(s) = \omega(s), \, s < t, \) and \( \Delta \) elsewhere.

**DEFINITION 2.1.** A family \( \{L^t, t \geq 0\} \) of \( \mathcal{F} \)-measurable functions will be called a coterminal family if it satisfies (i) \( 0 \leq L^t \circ k_t \leq L^t \leq t \), (ii) \( L^t \circ \theta_s = (L^t - s)^+ \) for \( s < t \), where \( a^+ = a \lor 0 \), (iii) \( \lim_{u \downarrow s} L^t \circ k_u = L^s \) for \( s < t \), and (iv) \( L^t < s < t \) implies \( L^t = L^s \).

**DEFINITION 2.2.** \( L \) is an exact coterminal time if it is \( \mathcal{F} \)-measurable and satisfies (i) \( 0 \leq L \circ k_t \leq t \), (ii) \( L \circ \theta_s = (L - s)^+ \), (iii) \( L \circ k_s = L \) on \( \{L < s\} \) and (iv) \( L = \lim_{s \to \infty} L \circ k_s \).

To avoid technical difficulties we have avoided a.s. statements and have assumed both definitions hold on all of \( \Omega \).

Coterminal times are studied in [3], and our times \( L^t \) can be defined from an exact coterminal time by \( L^t = \lim_{u \uparrow t} L \circ k^t \). Conversely, \( L \equiv \lim_{t \uparrow \infty} L^t \) is an exact coterminal time. Moreover, as in [3], we have

**LEMMA 2.1.** There exists a perfect, exact terminal time \( T \) associated with \( \{L^t, t \geq 0\} \) such that

\[ T = \inf\{s: L^s > 0\}. \]

Alternatively it is possible to begin with such a terminal time and define the family \( \{L^t, t \geq 0\} \) by

\[ L^t = \lim_{v \uparrow t} \sup_u \{u: T \circ \theta_u \circ k_v < \infty\}. \]

The duality between \( T \) and \( \{L^t, t \geq 0\} \) is canonical in the sense that \( T \) can then be retrieved via equation (4).

3. The \( \sigma \)-fields we use are given in

**DEFINITION 3.1.** The past of a random variable \( R \) is the \( \sigma \)-field \( \mathcal{F}(R-) \) generated by sets \( E_s \cap \{s < R\}, \, 0 \leq s < \infty \). The past plus present of \( R \) is the \( \sigma \)-field \( \mathcal{F}(R) \) generated by \( \mathcal{F}(R-) \) and \( \sigma(X_R) \).
These definitions together with an extensive discussion are given in [2].

4. Let \( \{L', t \geq 0\} \) be a fixed coterminal family and \( T \) the associated terminal time.

**Definition 4.1.** Let \( A' = \{x : P^x(T = 0) = 1\} \) and \( A^l = \{x : P^x(T = 0) = 0\} \).

One conditional distribution is given in

**Definition 4.2.** Let \( f \in b\delta_A \), the bounded universally measurable functions on \( E_A \). If \( x \in A^l \) and \( s > 0 \),

\[
D(x, s, f) = E^{x}[f(X_{s}) | T > s],
\]

with the convention throughout that \( 0/0 = 0 \). For \( x \in A' \) let

\[
D_n(x, u, s, f) = E^{x}[f(X_{s}) | \rho(x, X_u) < 1/n, T \land \theta_u > s - u].
\]

Then

\[
\bar{D}(x, s, f) = \lim_{n \to \infty} \lim_{u \to 0} D_n(x, u, s, f),
\]

and

\[
D(x, s, f) = \lim_{n} \lim_{u \to 0} D_n(x, u, s, f),
\]

where \( u \to 0 \) through the rationals. If \( \bar{D}(x, s, f) = D(x, s, f) \), we say \( D(x, s, f) \) exists and denotes the common value.

Then

**Theorem 1.** If \( f \in b\delta_A \), almost surely on \( \{0 \leq L' < t\} \), \( D(X_{L'}, t - L', f) \) exists and

\[
E[f(X_t) | \mathcal{F}(L')] = D(X_{L'}, t - L', f).
\]

5. Conditioning by \( \mathcal{F}(L' -) \) requires a different distribution. Let \( f, h \in b\delta_A \). Then

**Definition 5.1.** If \( x \in A^l \), \( Q^h(x, s, f) = E^x[h(x) \cdot f(X_s) | T > s] \). If \( x \in A' \), let \( Q^h(x, u, s, f) = E^x[h(X_u) \cdot f(X_s) | T \land \theta_u > s - u] \) and let \( \bar{Q}^h(x, s, f) \) (\( Q^h(x, s, f) \)) be the lim sup (lim inf) as \( u \to 0 \) through the rationals. If the two are equal, we say \( Q^h(x, s, f) \) exists and denotes the common value.

To prove Theorem 2 some control over left-hand limits is essential, and to simplify the presentation we assume henceforth that \( X \) is a standard process. We emphasize, however, that Theorem 1 and a version of Theorem 2 are valid when branching points are permitted.

**Theorem 2.** If \( f \in b\delta_A \) and \( h \) is bounded continuous, \( Q^h(X_{L' -}, t - L', f) \) exists almost surely on \( \{0 < L' < t\} \) and

\[
E[h(X_{L'}) \cdot f(X_t) | \mathcal{F}(L' -)] = Q^h(X_{L' -}, t - L', f).
\]
As is to be expected $Q^h$ and $D$ are related. In fact for both $f$ and $h \in b\mathcal{E}_\Delta$ we have

**Theorem 3.** There exists a family of probability measures $\{\eta(x, s, dz), (x, s) \in E_\Delta \times (0, \infty)\}$ such that, for $B \in \mathcal{B}_x, \eta(\cdot, \cdot, B)$ is universally measurable in the product space $(E_\Delta \times (0, \infty))$ and such that

$$E[h(X_{L'}), f(X_t)|\mathcal{F}(L' -)] = \int_{E_\Delta} \eta(X_{L', t - L'}, dz)D(z, t - L', f)h(z)$$

almost surely on $\{0 < L' < t\}$.

6. If $\mathcal{H}$ is a countable collection of functions in $b\mathcal{E}_\Delta$, then by Theorem 1 almost surely on $\{0 \leq L' < t\}$

$$(X_{L'}, t - L') \in \bigcap_{f \in \mathcal{H}} \{(x, s): D(x, s, f) \text{ exists}\}.$$  

Choosing an appropriate $\mathcal{H}$ including a dense subset of the continuous functions on $E_\Delta$, we obtain for each $(x, s)$ in the intersection above a Borel measure $D(x, s, \cdot)$ such that

$$D(x, s, f) = \int D(x, s, dy)f(y)$$

for all $f \in \mathcal{H}$. A set $G_D \subset E_\Delta$ is then defined such that for each $x \in G_D$ the measures $D(x, s, \cdot)$ can be extended to all $s > 0$ and normalizing parameters $d(x, s)$ defined so that $\{D(x, s, \cdot)/d(x, s), 0 < s < \infty\}$ form an entrance law relative to the family of transition probabilities $H_t(x, f) = E[X_{T > t}; f(X_t)]$. In addition $X_{L'} \in G_D$ a.s. $P^\mu$ for each $\mu$ and $t > 0$ on $\{0 \leq L' < t\}$.

An identical analysis holds for $Q^h$, and we have completed the analogy with (3) above. Moreover we can go one step further and use the entrance law property to prove the following version of the strong Markov property for $L'$.

**Theorem 4.** Let $f_i \subset b\mathcal{E}_\Delta$, $1 \leq i \leq n + 1$, and $0 < s_1 < \cdots < s_n < t$. If $\tilde{t}(\omega) \equiv t - L'(\omega)$, then almost surely, on $\{0 \leq L' < t - s_n\}$,

$$E[f_n+1(X_t) \prod_{k=1}^{n} f_k(X_{L'+s_k})|\mathcal{F}(L')]$$

$$= \int \cdots \int \frac{D(X_{L'}, s_1, dz)}{D(X_{L'}, s_1, H_{t-s_1})} f_1(z_1)H_{s_2-s_1}(z_1, dz_2) \cdots f_n(z_n)H_{t-s_n}(z_n, f_{n+1})$$

If $\mathcal{F}(L')$ is replaced by $\mathcal{F}(L' -)$, a similar expression holds with $D$ replaced by $Q^1$ and $X_{L' -}$ by $X_{L'}$. 
7. In the event \( \{L', t \geq 0\} \) is generated by a nontrivial exact coterminal time \( L \), all of the foregoing results can be extended to \( L \). Two families of probability distributions \( Q^h_L \) and \( D_L \) can be shown to exist and to satisfy the analogues of Theorems 1, 2 and 3. As in §6 both families define entrance laws, here relative to

\[
K_t(x, f) = E^x[f(X_t)|T = \infty].
\]

Furthermore that property can be used to establish a strong Markov property of the form given in Theorem 4 with \( t = \infty \), and \( H \) replaced by \( K \).

REFERENCES

