BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

NONANALYTIC-HYPOELLIPTICITY FOR SOME DEGENERATE ELLIPTIC OPERATORS

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Communicated by J. J. Kohn, October 21, 1971

We give here an example, as simple as possible, of a degenerate elliptic operator $\sum_{j=1}^{r} X_j^2$ where $X_1, X_2, \ldots, X_r$ are r vector fields with analytic coefficients which, with their commutators of order 1, span the whole space, and such that there exists a nonanalytic function u in the Gevrey class G_2 with $\sum_{j=1}^{r} X_j^2 u = 0$.

1. We consider an operator

$$A = yP + Q \tag{1}$$

where P is a second order elliptic (nondegenerate) operator and Q is a first order operator; we assume the coefficients of P and Q are analytic in some neighborhood $\mathcal{O}$ of the origin in $\boldsymbol{R}^n = \{(x, y);\ x \in \boldsymbol{R}^{n-1} \text{ and } y \in \boldsymbol{R}\}$. For simplicity we suppose $[P, Q] = PQ - QP = 0$ (however it is possible to consider more general situations). We assume $n > 1$.

We obtain the following result:

PROPOSITION 1. *Let V be a neighborhood of the origin in $\boldsymbol{R}^{n-1} \times \overline{\boldsymbol{R}}_+ = \{(x, y);\ x \in \boldsymbol{R}^{n-1}$ and $y \in [0, \infty[\}$ which is relatively compact in $\mathcal{O}$. There exists a function $u \in G_2(\overline{V})$,[2] whose restriction to any neighborhood of the origin is nonanalytic, such that there exists a constant $C > 0$ with*

$$\|D^\alpha A^k u\|_{L^2(V)} \leqq C^{|\alpha|+k+1}(2k)!(2\alpha)! \tag{2}$$

for each $k \in \boldsymbol{N}$ and $\alpha \in \boldsymbol{N}^n$.

PROOF. We note $\Gamma = \overline{\mathcal{O}} \cap \{(x, y) \in \boldsymbol{R}^{n-1} \times \boldsymbol{R};\ y = 0\}$. Let g be in $G_2(\Gamma)$ and nonanalytic in any neighborhood of the origin in $\boldsymbol{R}^{n-1}$. We construct a function u in some neighborhood of $\overline{V}$ in $\boldsymbol{R}^{n-1} \times \overline{\boldsymbol{R}}_+$ such that

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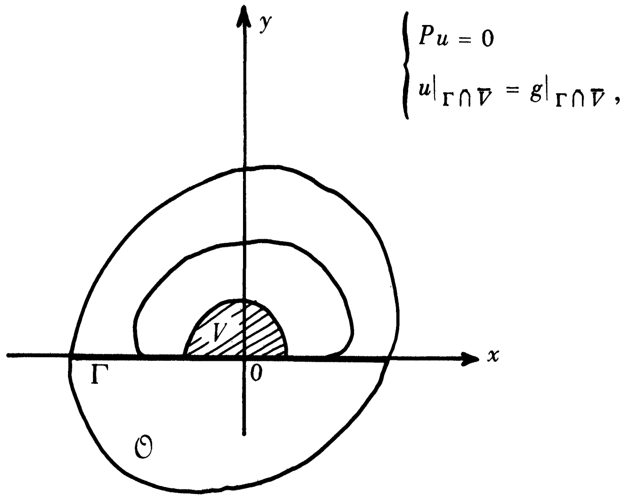
AMS 1970 *subject classifications.* Primary 35H05, 35J70; Secondary 35A05.
Key words and phrases. Hypoellipticity, analytic-hypoellipticity, degenerate elliptic operators.

[1] This research was partially done when the second author was visiting professor in Purdue University.

[2] The space $G_2(\overline{V})$ is the Gevrey space of order 2 which consists of functions $v \in \mathscr{C}^\infty(\overline{V})$ such that there exists a constant $C > 0$ with $\|D^\alpha v\|_{L^2(V)} \leqq C^{|\alpha|+1}(2\alpha)!$ for each $\alpha \in \boldsymbol{N}^n$.



by solving a Dirichlet problem in some neighborhood of $\bar{V}$ in $\boldsymbol{R}^{n-1} \times \bar{\boldsymbol{R}}_+$. Then $u \in G_2(\bar{V})$ and is nonanalytic in any neighborhood of the origin (see [**4**]).

We get obviously

$$A^k u = Q^k u \quad \text{in } V.$$

Now the proof can be completed using the following result:

For each $v \in G_2(\bar{V})$, there exists a constant $C > 0$ such that, for every $k \in \boldsymbol{N}$ and $\alpha \in \boldsymbol{N}^n$,

$$\|D^\alpha Q^k v\|_{L^2(V)} \leqq C^{|\alpha|+k+1}(2\alpha)!(2k)!.$$

2. We consider the operator

$$B = A + D_t^2 = yP + Q + D_t^2, \tag{3}$$ [3]

in the neighborhood $\mathcal{O} \times \boldsymbol{R}$ in $\boldsymbol{R}^{n+1} = \{(x, y, t); x \in \boldsymbol{R}^{n-1}, y \in \boldsymbol{R}, t \in \boldsymbol{R}\}$.

We have the following result:

PROPOSITION 2. *There exists a neighborhood W of the origin in $\boldsymbol{R}^{n-1} \times \bar{\boldsymbol{R}}_+ \times \boldsymbol{R}$ and a function $w \in G_2(\bar{W})$ whose restriction to any neighborhood of the origin is not analytic, such that*

$$Bw = 0 \quad \textit{in } W.$$

PROOF. Let us consider the series

$$w(x, y, t) = \sum_{m=0}^{\infty} t^{2m} \frac{A^m u(x, y)}{(2m)!}, \tag{4}$$ [4]

[3] We denote by D_t the operator $-i\partial/\partial t$.

[4] Such a series is also used in [**4**].

where u is given by Proposition 1. By using (2) it is easily seen that the function w is defined in $W = V \times [-\delta, +\delta]$ where δ is some suitable strictly positive number, and satisfies

$$Bw = 0 \quad \text{in } W$$

and there exists $M > 0$ such that

$$\|D^{\alpha}_{x,y} D^k_t w\|_{L^2(W)} \leqq M^{|\alpha|+k+1} k!(2\alpha)!$$

for each $k \in \boldsymbol{N}$ and $\alpha \in \boldsymbol{N}^n$.

Furthermore we have

$$w(x, y, 0) = u(x, y);$$

then w is nonanalytic in any neighborhood of the origin.

3. **Examples and applications.** Let us consider, for example, the following simple case (with $n = 2$):

$$P = D_x^2 + 4D_y^2,$$
$$Q = -2miD_y \quad \text{with } m \text{ integer} \geqq 1.$$

Then

$$B = y(D_x^2 + 4D_y^2) - 2miD_y + D_t^2. \tag{5}$$

We use the change of variables

$$y = z_1^2 + \cdots + z_m^2. \tag{6}$$

We denote $\tilde{w}$ by

$$\tilde{w}(x, z_1, \ldots, z_m, t) = w(x, z_1^2 + \cdots + z_m^2, t)$$

where w is given by Proposition 2.

The function $\tilde{w}$ is in the Gevrey class of order 2 in some neighborhood of the origin in $\boldsymbol{R}^{m+2}$ and nonanalytic. (If $\tilde{w}$ were analytic, the function $(x, z_1, t) \mapsto w(x, z_1^2, t)$ would be analytic too in some neighborhood of the origin in $\boldsymbol{R}^3$. The latter function is even with respect to z_1, so the function w would be also analytic in some neighborhood of the origin in $\boldsymbol{R} \times \overline{\boldsymbol{R}}_+ \times \boldsymbol{R}$, which contradicts Proposition 2.)

By the change of variables (6), the operator B defined by (5) becomes

$$H = (z_1^2 + \cdots + z_m^2)D_x^2 + D_{z_1}^2 + \cdots + D_{z_m}^2 + D_t^2$$

which can be written also in the form

$$H = \sum_{j=1}^{m} (z_j D_x)^2 + \sum_{j=1}^{m} D_{z_j}^2 + D_t^2. \tag{7}$$

In some neighborhood of the origin in $\boldsymbol{R}^{m+2}$ we have $H\tilde{w} = 0$. Therefore the following result is proved:

THEOREM. *Let m be an integer* $\geqq 1$. *The following operator*

$$H = \sum_{j=1}^{m} (z_j D_x)^2 + \sum_{j=1}^{m} D_{z_j}^2 + D_t^2$$

is not analytic-hypoelliptic in $\boldsymbol{R}^{m+2}$. *More precisely, one can find a function* $\tilde{w}$ *defined in some neighborhood of the origin, belonging to the Gevrey class of order* 2, *nonanalytic and such that* $H\tilde{w} = 0$.

In fact, we can construct, by the same method used here, a function $\tilde{w}$ which does not belong to any Gevrey class of order $\varepsilon < 2$ and which satisfies $H\tilde{w} = 0$.

The operator H is obviously of the form $\sum X_j^2$ and satisfies the Hörmander condition (see [**3**]), namely in this case the vector fields X_j and their commutators of order 1 span the whole space.

If, in the example (7), we take $m = 1$, it turns out that the operator $z^2 D_x^2 + D_z^2 + D_t^2$ is not analytic-hypoelliptic in $\boldsymbol{R}^3$; but it is known (see [**5**]) that the operator

$$z^2 D_x^2 + D_z^2 \tag{8}$$

is analytic-hypoelliptic in $\boldsymbol{R}^2$. Let us point out that M. Derridj and C. Zuily have also announced recently the analytic-hypoellipticity for same classes of operators which can be considered as generalizations of (8).

On the other hand, Proposition 2 gives a negative result of analyticity up to the boundary; positive results were given in [**1**], [**2**] for some classes of degenerate elliptic operators apparently not far from those of Propositions 1 and 2.

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